Geometry

Spirit and Methods

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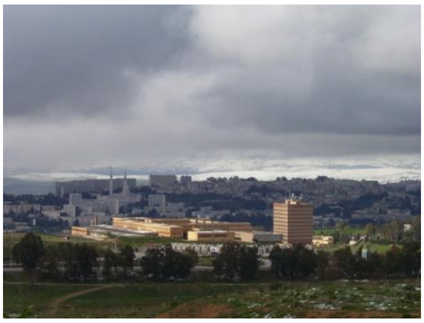

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The book still being updated, any mistakes occurring here are mine.

On the front image the reader can see a metal object in the form of a mollusk called *nautilus*. On nautilus' shell is engraved a natural curve from Antiquity, the *logarithmic spiral* whose polar equation is $\rho = a^{\varphi}$, it is a relative to the *Archimedes spiral*.

Preface

Geometry is perhaps the oldest branch of mathematics and it is surely now the most expanding science. It was really the essence of classical mathematical sciences, in such a way that not very long ago, a *mathematician* was still called a *geometer*. One of the very best features of geometry is that it may be taught at an early age, as a first hint to learn mathematical reasoning and what a proof is.

Both as a practical and theoretical science, Geometry is a very fascinating subject that, after a long time of stagnation, has departed from classical simple studies to delve into the realm of analysis, theoretical physics, number theory,… it has not stopped to give new problems for research, and with old vulnerable unsolved ones that still challenging and wait for mathematicians to solve them.

Geometry has before our times challenged the greatest minds to solve the four problems of Antiquity which have been solved only in the XIX century :

i) Trisection of an arbitrary angle.

ii) Squaring a circle.

iii) Doubling a cube.

iv) Constructing all the regular polygons.

During the XIX century, geometry has seen a dramatic shift from the classical synthetic and analytic features, to the use of new methods, essentially algebraic.

Among those big transformations that geometers have seen in their discipline, was what we call: *the second*[1] *algebraization of geometry*: The implementation of abstract algebra in geometrical studies that has created new still fascinating branches: algebraic geometry, algebraic topology,… The classical methods, which were the heart of classical geometry and have seen great preoccupations from scholars through centuries, are today taught only in college and high school.

My aim in this book is to expose geometry ideas to a student who intends, to specialize in this important part of mathematics, introducing him into research. This is not an elementary course; he will not learn geometry from nothing. I have tried to gather quite independent chapters on this science, which would have one aim: to teach the student what geometry is, and learn some of its methods. So he is supposed to be acquainted with geometry of high school for example, or he is, at least, strong in middle school geometry and perhaps has just encountered new methods in university.

And moreover, this is not a history of geometry either, because I am not specialist in this domain, but despite it is not a historical book, I preferred to follow the historical frame as far as possible, often with many digressions. I think that the historical order will give the students an easy understanding, of how geometry ideas developed through centuries of efforts by intelligent people, why it looks like this today. Very importantly, how it has intertwined with other mathematical disciplines. I am not responsible of the problem of the priority of the discoveries; the question does not concern me here.

Although I tried to study the most important geometric studies in independent chapters, I could not encompass all the subjects, but the reader will find in the text allusion to some other geometrical subjects like

Graph theory, synthetic geometry, metric geometry, finite geometry, geometric analysis, topological algebra, geometric algebra, geometric topology, arithmetic geometry, **Diophantine** geometry, descriptive geometry

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[1] The first algebraization has been accomplished by **Descartes** when he made a fusion between classical geometry and classical algebra.

I think that this book can be read by any layman with a mathematical culture, I have simplified it as much as I could. The most important part of it to my eyes is the exercises, which I vividly recommend to the reader, and I recommend to him to participate actively by doing all calculations and drawings. I will be a very happy if the student who has loved geometry to find the book interesting and serving.

Contents

Early geometry: …………………………………………………………………………….. 11

Some logic : ……………………………………………………………………………………………..11
Line and segment : ……………………………………………………………………………………..13
Polygons : ………………………………………………………………………………………………..14
Angles : ……………………………………………………………………………………………………15
Trigonometric functions:……………………………………………………………………… ………16
Solids and polyhedra:……………………………………………………………………………………..17
Simplex:……………………………………………………………………………………………………..18
Convexity………………………………………………………………………………………………….19

Measurements : ………………………………………………………………………………..30

Distance:………………………………………………………………………………………………….23
Triangle area:……………………………………………………………………………………………..23
The magic π:………………………………………………………………………………………………24
Radian:……………………………………………………………………………………………………..26
The cone revisited:………………………………………………………………………………………..28
Circle and sphere:…………………………………………………………………………………………30
Parabola:…………………………………………………………………………………………………..32
Fermat and Pascal:………………………………………………………………………………………..36
Roberval and Cavalieri:……………………………………………………………………………………40

Construction of plan figures : …………………………………………………………..42

Introducing geometric tools :…………………………………………………………………………….42
Abstraction problems:……………………………………………………………………………………..44
construction of rational numbers: ……………………………………………………………………..46

Construction by revolution:……………………………………………………48
Pappus-Guldin theorems :……………………………………………………52

Geometry as a theory : ……………………………………………………57

Structure of a geometric theory:……………………………………………58
Methods of proof:……………………………………………………………59
Thales:………………………………………………………………………61
Pythagoras theorem:…………………………………………………………63
Incommensurable numbers:……………………………………………………65

The Elements of Euclid ……………………………………………………70

Definition:……………………………………………………………………73
Postulates:……………………………………………………………………75
Axioms:………………………………………………………………………77
The fifth axiom:………………………………………………………………80
theorems:……………………………………………………………………81
the fifth Grail:…………………………………………………………………83

Euclidian flaws: ……………………………………………………………85

Archimedes:…………………………………………………………………87
Axiomatic method:……………………………………………………………89
what is an axiom………………………………………………………………90

The fifth axiom: ……………………………………………………………92

What is wrong with the 5th axiom ……………………………………………96
Hilbert axioms:………………………………………………………………98

Euclid legacy : …………………………………………………………… 100
Apollonius and the conics:……………………………………………………103
Fermat and the others:…………………………………………………………107

1

Early Geometry

Geometry is not a very bizarre activity for human beings, its beginning was with their contemplation of shapes around their environment; humans certainly have been attracted by *perfect* shapes: the shape of a square, of a circle, the shape of the full moon, the right angle,…

Before transforming it to a deductive science as we will see, geometry was born as a practical art. Babylonians geometry was not very developed; it was rather implemented in practices to build buildings and in the construction of worship monuments…. The inherited part of these studies was *descriptive geometry* founded mainly by French Gaspard **Monge** (1746-1818) as a tool for engineering; its aim was to *describe* geometric figures, as **Monge** defined it in his book

> *Descriptive geometry has two objectives: the first to give the methods to represent on a sheet of paper of two dimensions all the objects of nature that have three dimensions… the second is to give the methods to reconstruct these objects according to these description …*

For early Egyptians, geometry activity had for aim to calculate surfaces, because they used to calculate the areas of the peasants' lands and crops near the Nil River in order to evaluate taxes. The Greeks have been the real masters of this subject and almost all the heritage from antiquity were their creation. The word geometry itself comes from two Greek words: $\gamma\varepsilon\omega$ which means *earth*, and $\mu\varepsilon\tau\rho\omega$ which means *measure*.

However Greek mathematicians among them **Thales**, who is considered as the most ancient Greek geometer, have learned much from both Babylonians and Egyptians.

Some logic

Any scientist doing his work, needs reasoning rules called *Logic* which prevents him to fell in contradictions. Our logic that used in mathematics in general, belongs to any one that has the *common sense*. Among these rules that have been used by ancient thinkers in their investigations, not only in geometry, but in all aspects of life are :

i) Any statement A is either true or false; we present this by logic symbols

$$A \vee \neg A$$

There are complex statements that do not enter in this, because we do not know the truth about them if they are true or false like the *continuum hypothesis*. Other sentences are not simple, and to be analyzed like if one says : " an electron is neither a particle nor a wave" , other are mistakefully asserted: “the *number 1 is neither prime nor composite*” , here “ the number 1” is not a statement (a proposition).

ii) Any two quantities that are equal to another are equal

$$(A = B) \wedge (B = C) \Longrightarrow A = C$$

iii) A quantity is greater than any of its parts (remark that is not true in *Cantor theory of cardinals*, but it does not concern us here since we are studying finite quantities).

iv) The implication " $\Longrightarrow$ " is the most important ingredient in mathematics proof. One should be more accurate , usually we encounter problems if we state some sentences related to the implication. For example the sentence

$f'' \geq 0$, *whenever the function* f is convex

is not accurate because we do not know the relation between "$f'' < 0$" and *"the function* f is convex" , to be more accurate we may say

If $f'' \geq 0$ *then (or* $\Longrightarrow$*) the function* f is convex

Then automatically we search for more accurate information, and we look for the inverse(reciprocal) statement, this lead us to the equivalence, (sufficient and necessary conditions) which is very important in mathematics ,we wish to have for example

The function f *is convex* $\Longleftrightarrow$ *(iff)* $f'' \geq 0$

As an example very likely known to the reader

$$\Delta \llcorner (\Delta_1 \parallel \Delta_2) \Longleftrightarrow \mathit{all\ exterior\ \ angels\ are\ equal}$$

Unfortunately we cannot always find (and prove) equivalences between mathematical statements.

Line and segment

Scientists, whatever the subject of their science is, have an agreement that humans cannot *define* all things. For example in physics, physicists have not succeeded in finding definitions for three important concepts of physics, namely:

Mass, force, energy.

Likewise since a long time, mathematicians could not define *mathematically* many things in geometry like

Point, line, length, surface …

without using words from outside mathematics. Then if we want here to develop our science, we must sketch a *mathematical definition* by using intuition and words to approach their meanings[2].

$i)$ A *line* is a *straight* set of points in $\mathbb{R}^2$ or $\mathbb{R}^3$, that have no ends in the two directions; we may note a line by the symbol

$$\Delta$$

but a common *general line* can be either straight or curved. Two different lines Δ_1 and Δ_2 are *parallel* if they do not intersect, a propriety noted by

$$\Delta_1 \parallel \Delta_2$$

$ii)$ A *segment* (or line segment) is a finite portion of a line, we note a segment between two points A and B by

$$\overline{AB}$$

---

[2] And it was **Euclid** who first did this as we will see.

The straight *length* between A and B, or the length of the segment $\overline{AB}$ (with a chosen *length unit*) is simply noted

$$AB$$

The *median* of a segment is a line that divides it in two equal parts. On a line, the median can pass through any point

Until later chapters, we will keep doing our geometric investigations in our natural environment, so we work in:

i) One dimension: presented by a line, or simply by the real line $\mathbb{R}$.

ii) Two dimensions: it is the known plan; we can for example draw plan figures in $\mathbb{R}^2$.

iii) Three dimensions: the very natural place where we live and is presented mathematically by $\mathbb{R}^3$.

Polygons

Definition:

A *polygon* is a figure in the plane $\mathbb{R}^2$ (or in $\mathbb{R}^3$) that has more than two straight sides, every two sides (or *edges*) are linked by a point called *vertex*.

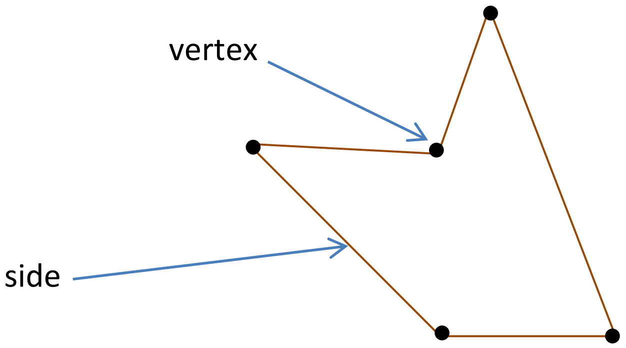

Fig. A polygon with 5 sides

The most common polygons are

i) Triangle: a polygon with three sides, a triangle with vertices A, B and C is usually noted by the symbol

$$\Delta ABC$$

A triangle with the lengths a, b and c is written

$$\Delta(a, b, c)$$

A triangle ABC can be *isosceles* (two same sides' length), *equilateral* (all same sides' length), or *right* (with a right angle).

A triangle, as many other figures, has some curious properties : a line passing by the middles of two sides is parallel to the other side and its length is $\frac{1}{2}$ of its length (see : ***Thales** proportions*). There are many special lines that meet in one point inside the triangle.

ii) Rectangle: a polygon with four sides, and four right angles.

iii) Square: it is a special rectangle where all the four sides are equal. Having 4 sides, rectangles, squares,… are all *quadrilaterals.* They are also called *parallelograms:* their sides are parallel two by two.

iv) Rhombus: it is a special quadrilateral parallelogram, it is not necessary rectangle.

Angles

An angle is the stretch between two sides (segments, lines or *rays*), and is noted by the symbols

$$\angle ABC \qquad \text{or} \qquad \widehat{ABC}$$

where B is the angle vertex. Both notations are boring, for an easy reading, it is better to note an angle only by its vertex

$$\hat{B}$$

We give a measure to an angle by the symbol

$$\measuredangle\hat{B}$$

or simply

$$\hat{B}$$

We will in this book use Greek letters for angles, the most used are

$$\alpha, \qquad \beta, \qquad \theta$$

The first task is to make a comparison between angles inside polygons. To confiscate a unit for measuring angles, we may divide a circle into 500 , 100 ,...(or any other number) equal parts. These are only conventions between mathematicians to facilitate the study of geometry. Anyway, the two common used units are :

1) *Degree*

Noted (°) , which has been inherited from the system of the Babylonians, the circle is divided in 360 equal parts i.e. 360^o , and we can divide the degree to other subunits as follows.

$i)$ *Minute* noted $(')$, we have 1 degree $=$ 60 minutes.

$ii)$ *Second*, noted (") , 1 minutes $=$ 60 seconds.

A line is regarded as an angle of 180° . The right angle measures 90° , if two lines or segments Δ_1 and Δ_2 intersect at an angle of 90° , they are called *perpendicular* or *orthogonal* lines and we note this relation between them by

$$\Delta_1 \perp \Delta_2 \qquad \text{(or } \Delta_1 \llcorner \Delta_2 \text{)}$$

Two angles α and β, whose measure is

$$\measuredangle(\alpha+\beta) = 90^o$$

are called *complementary* angles, and two angles α and β whose measure is

$$\measuredangle(\alpha+\beta) = 180^o$$

are *supplementary* angles. If two lines intersect at a point, every two *vertical*(or *opposite) angles* are equal.

We also can define three different angles :

$i)$ an angle α is *acute* if

==

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$$0° < \measuredangle\alpha < 90°$$

$ii)$ an angle α is ***obtuse*** if

$$90^o < \measuredangle\alpha < 180^o$$

$iii)$ an angle α is ***reflex*** if

$$180° < \measuredangle\alpha < 360°$$

The other unit used in measuring angles is :

2) Gradian

Noted (go), in this case the circle is divided in 400 equal parts i.e. 400 go, the gradian is still used in topography (the study of surfaces' shapes) and geodesy (the study of the Earth's shape). We have from this convention:

Theorem

If we choose the *degree* as a unit for measuring angles, than the sum of angles' measures in any *triangle* is 180^0 or $200\ go$.

Trigonometric functions:

As **Playfair** said in his book on Geometry(1804):

> *Trigonometry is defined in this text to be the application of number to express the relations of the sides and angles of triangles.*

It is a fact learned by experience, that we can *characterize* angles in a triangle only if one of its angles is right :

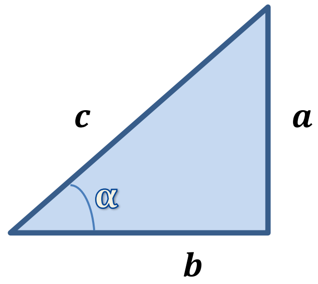

Fig. Right triangle

We can in a right triangle, define *circular* or *trigonometric* proprieties for an angle α by introducing two numbers

$$sin(\alpha) = \frac{a}{c}$$

and

$$cos(\alpha) = \frac{b}{c}$$

The segment with length c is the *hypotenuse* of the right triangle. We infer from a propriety of the right triangle, namely

$$\mathrm{c} > \quad a, b$$

that the values of $\sin(\alpha)$ and $\cos(\alpha)$ cannot exceed the value 1 :

$$0 < \sin(\alpha) < 1$$

and

$$0 < \cos(\alpha) \leq 1$$

also we can deduce that

$$sin(\alpha) = cos(90 - \alpha)$$

and more other relations that the reader has learned from pre-university school. From the combinations of sin and cos, new characteristic numbers can be introduced like:

$$tan(\alpha) = \frac{sin(\alpha)}{cos(\alpha)} = \frac{a}{b} \quad , \qquad cotan(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{b}{a}$$

and we can generalize these to *obtuse angles* by introducing the trigonometric circle.

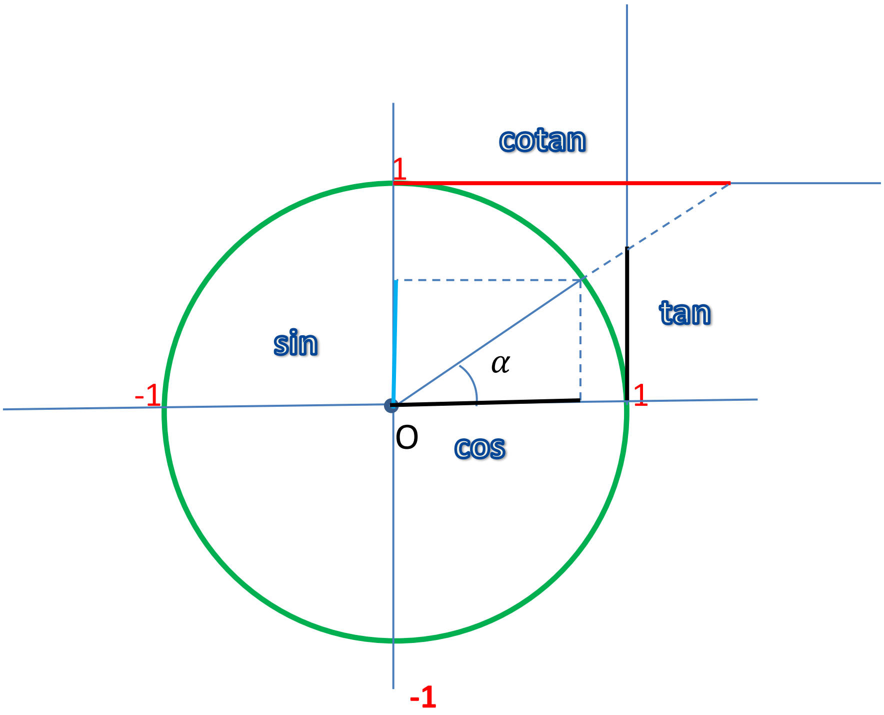

Fig. The trigonometric circle

When we introduce the real variable x in place of the angle α, we find the trigonometric or the *circular* functions and we have this time

$$-1 \leq \sin(x)\,, \cos(x) \leq 1$$

and

$$-\infty < \operatorname{tann}(x)\,, \operatorname{cotan}(x) < +\infty$$

For any triangle ABC we have two laws due in part to Muslim mathematicians who were very active during the Middle Ages:

Theorem

Law of $sins$

$$\frac{a}{sin(\alpha)} = \frac{b}{sin(\beta)} = \frac{c}{sin(\theta)}$$

Law of cos

$$a^2 = b^2 + c^2 - b \times c \times cos(\alpha)$$

The function cos is specially used in many constructions like projection (scalar product), directions,…

The Ancients were also preoccupied by the study of line segments in the circle, a circle is by the perimeter l and diameter d such that its radius R (or r) is :

$$radius = R = \frac{d}{2}$$

They particularly studied *chords* :

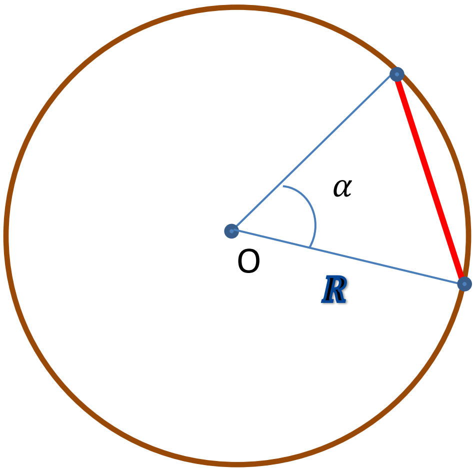

Fig. A *chord* in red.

and they were aware of formulas like

$$chord(2\alpha) = 2Rsin(\alpha)$$

Observe that the definition of trigonometric lines : $sin(\alpha)\,, cos(\alpha)$,... is linked to chords.

The study of trigonometry was much promoted by the French mathematician **Viète** (1540-1603) at the beginning of the Renaissance, who gave new formulas relation between

$$sin(\alpha)\,, \quad cos(\alpha), \dots$$

So, after we have considered polygons which are in fact "plan figures", we will see now another interesting part of geometry studies.

Solids and Polyhedra

So far we have been studying only plan figures, otherwise figures in the plane $\mathbb{R}^2$, and we can also imagine figures that fit in three dimensions $\mathbb{R}^3$, called *solid* figures, we will discover that plan figures are much easier to study. Until later chapters, we will be doing exclusively both *plan* and *solid geometry*.

A solid as a figure in space of three dimensions can have any form, but in order to do interesting geometry we focus our attention on some special ones. One interesting solid for example is the *ball*, a very important figure in geometry; a ball is surrounded by a three dimensional figure called a *sphere*[3] and noted S.

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[3] We will see in subsequent chapters that the sphere is noted more exactly S^2.

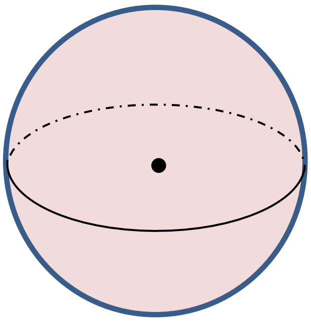

Fig. One should make a difference between a ***ball*** and a ***sphere***.

If the solid has only plane(or *flat*) faces it is called a *polyhedron*[4] , a very known and common polyhedron is the cube

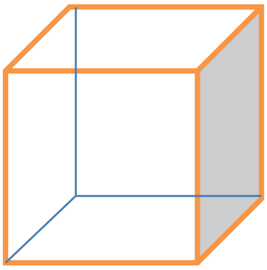

Fig. A *cube* is a polyhedron that has 6 plane or *flat* faces.

the smallest polyhedron is the *tetrahedron* which has 4 plan sides.

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[4] From Greek $\pi o\lambda \upsilon \varsigma$ "many" and $\varepsilon \delta \rho \alpha$ "base". The plural of *polyhedron* is usually *polyhedra*.

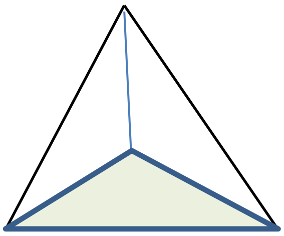


Fig. A *tetrahedron* is a polyhedron with 4 plane faces.

Ancient Greeks knew many polyhedral as:

$i)$ The pyramid.

$ii)$ The cube.

$iii)$ The dodecahedron (with 12 plane faces).

$iv)$ The icosidodecahedron (with 14 plane faces, by **Archimedes**).

$v)$ The icosahedron (with 20 plane faces).

They were also aware of a very amazing fact about these special solids, namely that there are only 5 regular (or perfect) polyhadra!! What does this mean? A polyhedron is *regular* or *perfect* if

all of its faces have the same shape and the same size.

The only 5 regular polyhedra are:

$i)$ the cube.
$ii)$ the dodecahedron.
$iii)$ the icosahedron.
$iv)$ the octahedron.
$v)$ the tetrahedron.

Another interesting solid is the *cone*

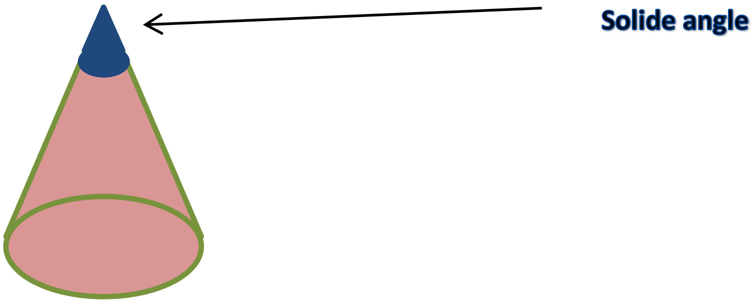


Fig. A *cone*.

From the cone rise the *conic sections* that we will see later. Precisely in this important solid, we can define a new concept: it is the *solid angle.*

Polyhedra have seen many interesting algebraic studies since the XIX century with the methods of algebraic topology, algebraic geometry,… and we now introduce an interesting polyhedron.

Simplex

A *simplex* is a polyhedron that has triangle-form faces, it is called also a *general tetrahedron*, and as a general figure k-simplex:

Definition

$i)$ A point is a 0-simplex

$ii)$ A line is a 1-simplex

$iii)$ A triangle a is 2-simplex

$iv)$ A pyramid is a 3-simplex

Simplexes are studied in algebraic topology and have very interesting theoretical proprieties used in optimization theory.

Convexity

It is a very important concept and has many uses in mathematics as analysis, optimization,… and here we consider this concept for geometric figures seen as sets of points , and is under the name of *convex sets* , convexity can be also defined for functions and abstract spaces,…

Definition

A plan figure F is *convex* if we can draw a line segment between any two points a and $\in F$, such that the entire line remains inside F.

By the same way, we also define convexity for solids in $\mathbb{R}^3$. French **Legendre** (1898) defined a convex polygon by saying *(p.4):*

We call convex polygon, a polygon situated entirely in the same place of the direction of its sides.

By eye, we see that triangles, parallelograms, simplexes…are convex figures. The inverse of convex is *concave*.

Exercises

In all the following exercises, do not use the **Pythagorean** Theorem.

1) Give the definition of the circle and the disc, by using words, then by using real numbers and complex numbers.
1) Should they be plan figures only?
1) Give their equations in $\mathbb{R}^3$.
1) Could you define them in $\mathbb{R}$?
1) What do I mean by : " On a line, the median can pass through any point "?
1) could we draw a triangle of arbitrary size $\Delta(a, b, c)$? what are the conditions?
1) find x in : $1^o = x\ go$? and $1go = x^{\,o}$?
1) what is the measure of a right angle in (go)?
1) say if the following statements are true or false:
i) all triangles are convex
ii) all quadrilaterals are convex
iii) in a triangle, the values of the angles always depend on the length of the sides
iv) a circle is the perimeter of a disc.
1) give four non convex polygons.
1) name all the special lines that meet all in one point inside a triangle
1) what figure that always has complementary angles? Supplementary ones?
1) tell why by definition the values of $cos(\theta)$ and $sin(\theta)$ cannot exceed 1
1) Show in a right triangle that $cos(\theta) = sin(90 - \theta)$. what is then $sin(\theta)$?
1) what are the values: $tan(90 - \theta)$, $cotan(90 - \theta)$?
1) may $sin(a)$ in the sin-law be negative? Why?
1) Give a figure in $\mathbb{R}^3$ that is not convex.
1) Let D_1, D_2 two different discs with centers O_1, O_2 , what is the geometric name of their intersection surface? How can you characterize it (definition)?
1) let D_1, D_2 be two similar disks with centers O_1, O_2 , give a definition to their intersection .

2

Measurements

The early main geometric studies were practical; the main task was to find the areas and volumes of figures and until very recent times, it was still one of geometers preoccupations, French mathematician ***Legendre*** *(1752-1833)* said *[p.5]:*

> *Geometry has for objective, the measure of figures' size and the study of their properties.*

Distance

A practical problem in geometry is to calculate the distance between two points A and B either in the plan or in the three dimensional space, which we noted $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Of course the value of a distance depends on the unit used to evaluate it. To be more general, geometers also define the distance between a point and a line:

Definition

To define the distance between p and the line Δ

$$d(p, \Delta)$$

we draw a line Δ' passing by p and perpendicular to Δ, then the distance between p and Δ is

$$d(p, \Delta) = pp'$$

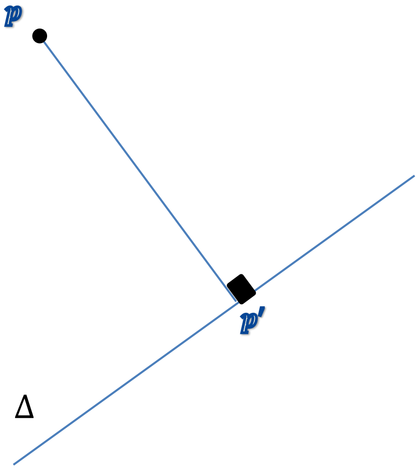


Fig. Distance between a point and a line.

Triangle area

An old method of calculating the area of a triangle ABC is the ***Hero*** *formula* ,due to mathematician **Hero** of Alexandria (10-70 AD) :

$$S = \sqrt{r(r-a)(r-b)(r-c)}$$

where $a = AB, b = \ BC, c = CA$, and

$$r = \frac{a+b+c}{2}$$

The magic π

One of the biggest things about geometric measurements that really worth mentioning, was the discovery of the number π, a magic gift from God for men to wonder about:

Theorem

The perimeter of any circle (or a disc) divided by its diameter is a constant number:

$$\frac{perimeter\ of\ a\ circle}{diameter} = \frac{l}{d} = \pi = 3{,}14\ldots$$

This number appears in a lot of mathematical contexts that apparently have nothing to do with geometry, like

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$\pi = -2i \times log(i)$$

The letter π was given to us by **Euler** in 1748. The Greek **Archimedes** succeeded in giving the approximation

$$3{,}112\ldots = \frac{221}{71} < \pi < \frac{22}{7} = 3{,}142\ldots$$

The first thing to know about that the number π is that it is not *perfect* like the circle; it is an *irrational* number and even worse, is a *transcendental* number. Many other numbers taken from *nature* show that it is more complex than one may think, besides we know from mathematical analysis that almost all the natural laws like those of movements,… are *nonlinear*.

Radian

The idea of *radian* as a unit of angles' measure is quite old and the name was introduced only in the XIX century, and exactly in 1870 by English physicist James **Thomson** (1822-1892).

It is natural that a geometer wants to get rid of any outside arbitrary chosen unity (like degree, gradian,…) to measure an angle. Since an angle α may always surrounded by an *arc of a circle* , which is linked to the radius r by the relation

$$\frac{arc\ length\ of\ \alpha}{r} = constant$$

then the idea is to link the angle to the length of its arc.
Thus the new unit ***1 radian*** is :

The measure of the angle surrounded by an arc whose length is equal to the radius

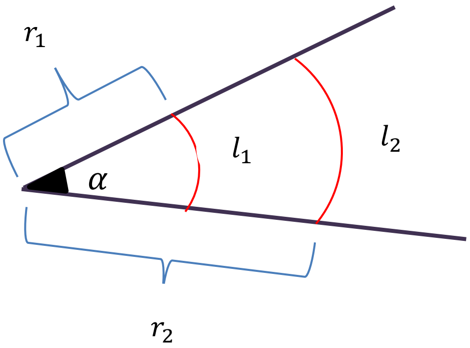


Fig. Under the same angle : $\frac{l_1}{r_1} = \frac{l_2}{r_2}$ is constant.

The aim behind that is to give an independent unit of measure. The new unit is *absolute* and does not depend on any exterior considerations, like the number π does not depend on the dimension of then circle.

Since the perimeter of a circle is $2\pi r$, the whole angle surrounded by a circle in the new unit is

$$\frac{2\pi r}{r} = 2\pi\ rad = 360^o = 400\ go$$

The reader now, sees that the measure of an angle is linked to π and is independent of any outside measurements; the radian is an *intrinsic* unit. This will also simplify the calculation of the circle perimeter if α is given in radian, we have then

$$\alpha r \qquad \text{versus} \qquad \frac{\alpha\pi r}{180}$$

and for a sector area, we have

$$\alpha r^2 \qquad \text{versus} \qquad \frac{\alpha \pi r^2}{360}$$

Moreover, the formulas for : $\sin(x)$,... in Analysis do not work for the units "degree" and "gradian" , but only for radian. As a classical example for have for sinus infinite series:

$$1 = sin\left(\frac{\pi}{2}\right) = \sum_{0}^{\infty}(-1)^n \frac{\left(\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} \neq sin(90) = \sum_{0}^{\infty}(-1)^n \frac{90^{2n+1}}{(2n+1)!} = 0{,}89 \ldots$$

This is simply because the infinite series for $sin(t)$ has been found by the mathematician **Gregory** (1638-1675) and others by considering exactly : *the arc length.*

The same is true for the infinite series of $arcsin(t)$, discovered by the English scientist Isaac **Newton** (1642-1727) around 1669 when he asked :

Question:

Knowing "$sin(x)$" , find the *arc length* "y".

Newton found that the arc length y is the area under the curve of the function $\frac{1}{\sqrt{1-x^2}}$, and by using the binomial theorem he found that

$$y = arcsin(x) = \sum_{0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \times \frac{x^{2n+1}}{2n+1}$$

==

the infinite series for $sin(x)$ can be extracted[5] from the previous series, and thus x is also in radian[6].

The cone revisited

The volume of a cone is

$$V_{cone} = \frac{2\pi Rh}{3}$$

and the magnitude of its solid angle is

$$\alpha = \frac{s}{r^2}$$

Where s is the surface of the angle basis, and r being the radius. The unit for measuring solid angles is the *steradian*[7].

Circle and Sphere

A circle C has a perimeter l that surrounds the surface[8] A_C

---

[5] See for this inversion: W. **Dunhan** , *The Calculus Gallery* , Princeton edition, p.17.

[6] Also in mechanics, the velocity of a point moving on any trajectory is $v = \frac{dx}{dt} = \frac{dR\theta}{dt} = R\frac{d\theta}{dt}$, where x is the trajectory length and θ is in radian. So the angular velocity $\omega = \frac{d\theta}{dt}$ is in radian per second.

[7] Fusion of "*square*" and "*radian*".

[8] By ordinary usage, geometers say "surface of a circle" and mean "surface of a disk", and likewise "volume of a sphere" and mean "volume of a ball".

Theorem:

Its *perimeter length* is

$$l = 2\pi r$$

surface area is

$$A_C = \pi r^2$$

If $S(r)$ is a sphere of radius r then

Theorem:

Its *surface area* is

$$A_s = 4\pi r^2$$

i.e. 4 times the area of a disk of the same radius r. The volume of the ball surrounded by a sphere is

$$V_s = \frac{4}{3}\pi r^3$$

In his book “*On the sphere and cylinder*” , of which the Latin manuscript still exists , **Archimedes** was the first to give the surface area of a sphere. The sphere is a geometric surface that will take much of our interest in the subsequent chapters.

On a sphere[9] we can define a solid angle as we have seen with the cone

---

[9] This surface, when on the sphere, can be a disk , a square…

$$\alpha = \frac{s}{r^2}$$

Another new thing about the sphere is that when we draw a triangle on it we find the sum of angles is bigger than 180^o , this lead us to define the *big circle*.

Definition:

A big circle in a sphere S is a circle resulting from the intersection of S and any plane that passes by its center.

Parabola

Archimedes also could calculate the area between a parabola and a line. He used what we call the *method of exhaustion*:

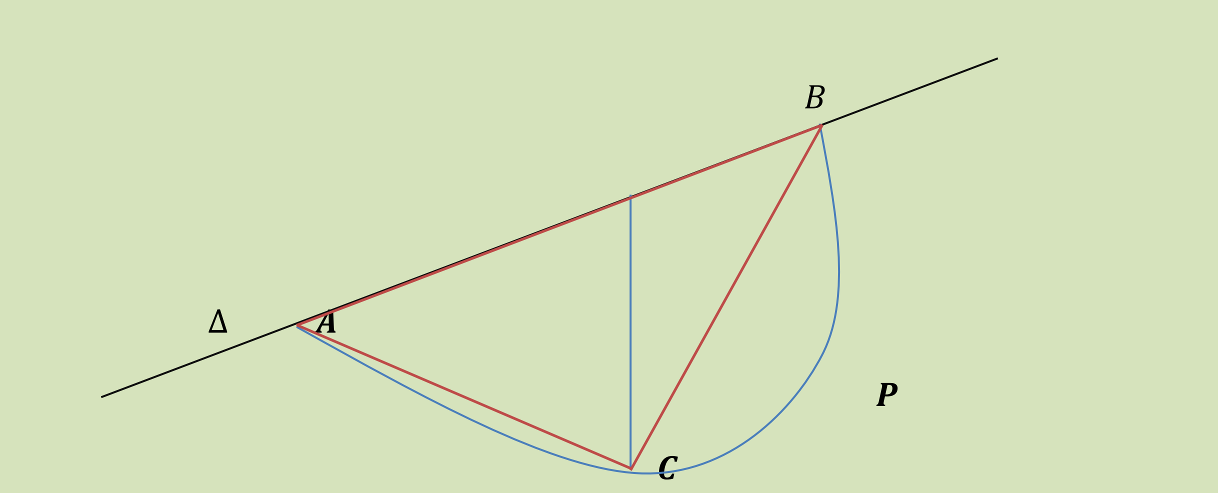


Fig. Triangles' surfaces to exhaust a part of a parabola $\boldsymbol{P}$.

This method consists in the following : Let G any figure we want to calculate the area, G is filled by figure whose area s_k is known, say triangles ABC_i , as small as we wish until its *exhaustion*, then the approximated area of G is the sum of all areas s_k

$$S = \sum_{1}^{N} s_k$$

the areas s_k are used in the form of a geometric sequence and for a given n the area is approximtly :

$$S \sim \left(1 - \left(\frac{1}{4}\right)^{n+1}\right) \times \frac{4}{3} \times area(ABC_i\ )$$

Fermat and **Pascal**

Both **Fermat** and **Pascal** used the same **Archimedes'** method of exhaustion but they renewed the method by founding what we now call the *exhaustion method of rectangles*, to calculate the area of a surface.

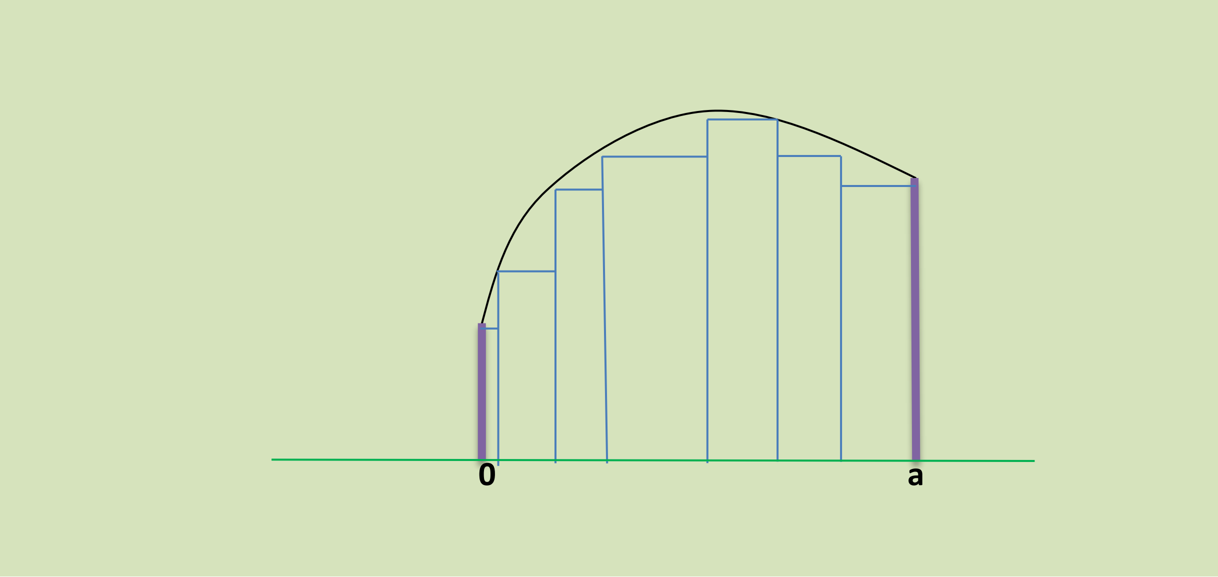


Fig. Exhausting the surface by rectangles.

To exhaust the parabola, **Fermat** (1601-1665) used rectangles of different widths. Suppose we want to evaluate the area of a portion of the parabola from abscissa 0 to a with rectangles rec_i of *different* widths r_i

$$s_k = t \times {r_i}^n$$

and all the r_i verify $0 < r_i < 1$ **,** so

$$t^3 \times r^2 \times \left(\frac{1-r}{1-r^3}\right) \leq S \leq t^3 \times \left(\frac{1-r}{1-r^3}\right)$$

Fermat considered as did **Archimedes**, the area as a limit :

$$S = \sum t \times {r_i}^n \approx \frac{t^3}{3}$$

In his book *Potestam numericarum summa* (1654), Blaise **Pascal** (1623-1662) chose the rectangles rec_i with the *same width* z,

$$s_k = t \times \frac{k}{n}$$

then this gives

$$z \times \sum_{1}^{N} s_k \leq S \leq \mathrm{z} \times \sum_{1}^{N} s_{k+1}$$

thus

$$S \approx \frac{a^3}{3}$$

Cavalieri and **Roberval**

In the 1640', the French **Roberval** (1602-1675) could calculate the area under the *cycloid*, a plane curve discovered (invented?) by **Galilei**.

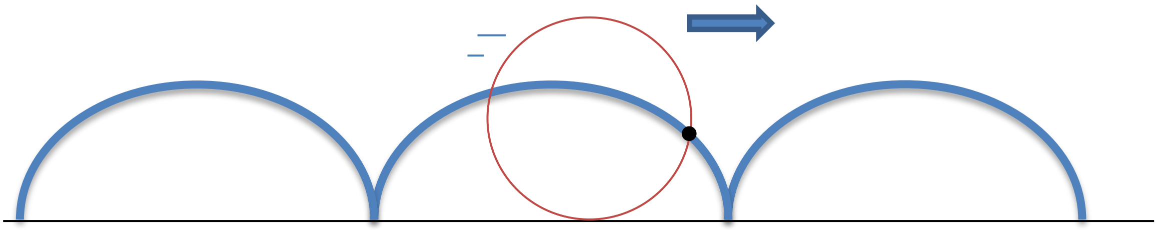


Fig. a cycloid curve in blue.

A *cycloid* is a plan figure that is drawn by the movement of a point fixed on a rotating circle without sliding. It can also be assimilated to the ***catenary***, the simple shape of a rope fixed at two pillars:

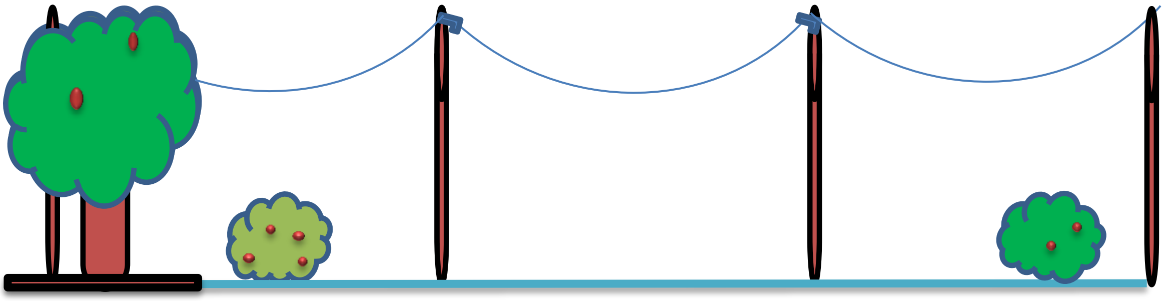


Fig. Catenary, inverse of a cycloid.

Around 1629, the Italian **Cavelieri**(1598-1647) in order to calculate areas , introduced a new method : he imagined that any surface is composed of the juxtaposition of infinitely many lines that he called ***indivisibles*** , however his method is not accurate and can lead to mistakes in the evaluation of areas.

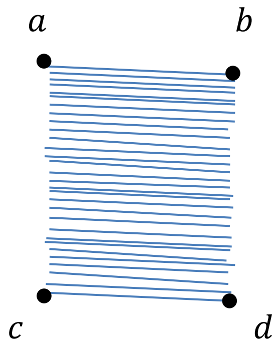


Fig. A sketch of **Cavalieri's** idea for a rectangle $abcd$.

Exercises

1) Why we have chosen the line in $d(p,\Delta)$ to be perpendicular?

1) Draw the triangle Δ(5,4,3). Calculate all its three heights : $h_1, h_2 \ and \ h_3$

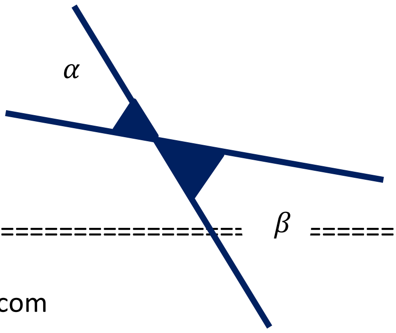
1) draw the triangle Δ(6,5,3). Calculate its perimeter. Find its area

1) draw the triangle Δ(7,5,3). Use the formula

$$S = \frac{ah_1 + bh_2 + ch_3}{6}$$

then the ***Heron*** *formula* . What do you remark?

1) show that *vertical* angles α and β , are equal:



1) what is the sum of the angles measure of a quadrilateral?

1) Calculate the perimeter and area of all forms of trapezium that you know

1) Calculate the perimeter and area of the rhombus of axis a.

1) Calculate the perimeter and area of the rhombus of axes a and b

1) Calculate the area and volume of a closed cylinder of height h and radius r

1) Calculate the area and volume of a closed inclined cylinder of side a and radius r

1) what the maximum value for and angle in a triangle?

1) some students think that the area of any parallelogram of sides a and b is $a \times b$. Show that it is true only for rectangles. (the **Cavalieri** method problem)

1) show that the area of a parallelogram is $a \times h$.

1) show that : $\frac{arc\ length\ of\ \alpha}{r} = constant$. Give the constant value.

1) the number π is an absolute constant linked to the circle, calculate the one linked to the square.

1) show that the areas of two circles are to each other like the squares of their radii.

1) let ABC a right triangle in A, and AD perpendicular on BC, show that the triangles ABD and ADC are similar

1) we put a sphere $S(r_1)$ of radius $r_1 = 5cm$ into a cylinder $C(r_2, l)$ of water , where $r_2 = 6\ cm$ and $l = 10cm$. Find the volume of water that pours out.

1) what is the volume of a regular and irregular pyramid? (**Archimedes** attributed the formula giving the volume of a pyramid to **Democritus**)

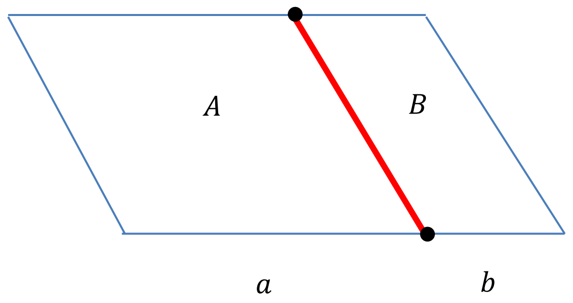
1) if a line intersects a circle in two point (a,b), show that is parallel to the diameter, calculate the length of the two arcs $\overset{\smile}{ab}$ and $\overset{\frown}{ab}$ then the length of the segment ab.

1) calculate the intersection area between two similar disks

1) The square and equilateral triangle are the two first regular polygons, name the third calculate its diagonal length.

1) if you have a regular n-polygon how can get a regular 2n-polygon (think of central angles)

1) let the areas (A, B) and lengths (a, b):



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show that : $\boldsymbol{\frac{A}{a}} = \boldsymbol{\frac{B}{b}}$.

1) in a triangle ABC the angle $\hat{A} = 45^0$ and $AB = 5$, $AC = 7$. could you find AC and the measures of the other angles .

1) if you know that the derivative of the function $arcsin(x)$ is $\frac{1}{\sqrt{1-x^2}}$, could you find out its infinite series?

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3

Construction of figures

Introducing geometric tools:

How can we construct geometric figures? For plane ones, there are some very old tools that have been used since the Ancient Greeks:

$i)$ *Ruler*: a tool used to draw straight lines, if it is not graded it is simply called a *straightedge*.

$ii)$ *Compass*: a tool used to draw circles and arcs (beware! the English word *compass* has two different meanings!).

$iii)$ *Protractor*: a practical tool to measure angles, it is a less theoretical tool.

With these tools at hand, we can draw a lot of geometric figures on a sheet of paper; these are very enjoying situations which all students have met already in pre-university schools. At the time we all were not aware of very good, interesting questions:

Question:

$i)$ Can we construct any plane figure by using a ruler and a compass only?

$ii)$ If the answer is "no", which figures could not be drawn?

Looking easy, these two questions have been the origin of some of the most difficult theoretical problems that needed centuries to be solved.

Abstract problems

No one can do mathematics without abstraction, it's an essential part of mathematical thinking and the concept of *"number* " was one of early and very fruitful abstractions. **Kline** said:

> *One of the great Greek contributions to the very concept of mathematics was the conscious recognition, and emphasis of the fact that mathematical entities are abstractions, ideas entertained by the mind and sharply distinguished from physical objects or pictures.*

So in the scrutiny manipulations by geometers raised many problems that first seemed only practical, and then turned into very difficult abstract problems. The search for solutions took mathematicians very far from geometry itself in new areas of pure mathematics.

Having our two ancient tools:

$*)$ A ruler to draw *lines*.

$*)$ A compass to draw *arcs*.

Then the question is : Can we by a ruler and a compass only, perform the following tasks

Questions

$i)$ Can we trisect an arbitrary angle?

$ii)$ Can we draw any perfect polygon of n sides?

These challenging problems had to wait for mathematicians of the XIX century:

$i)$ Impossible, proved by the French mathematician **Wantzel**(1814-1848) in 1837 and also by **Galois**(1811-1832) .

$ii)$ Possible if $n = 2^{2^p} + 1$, proved by **Gauss** in 1801.

We look now to another abstract problem studied much in the past, a problem that implied numbers worse than irrational numbers:

Question:

$iii)$ Draw a circle C, can we draw a square whose area is the area of the disk C?

This problem was given the name : *Quadrature of the circle* . It has been very influential in the invention of integral and differential calculus in the XVII century. It has been proven impossible by German mathematician **Lindeman**(1852-1939) by showing that the number π , the geometry's number by excellence, is *transcendental.*

Another problem looking like the quadrature of the circle, but cannot like the previous ones accomplished by a ruler and a compass, because it needs the three dimensions space:

$iv)$ Let C a cube of volume v_1 , can we get a cube of volume $v_2 = 2v_1$? (Doubling the cube). This problem is known as the *Delian problem*.

But although it is a problem in $\mathbb{R}^3$, it can be reduced to construct a segment r in the plane $\mathbb{R}^2$. It was proved also impossible by **Wantzel**(1837) and by **Galois**.

Construction of rational numbers

Some theorems in Greek mathematics belong today to the proprieties of real numbers and are not of the geometry. But Geometry has never been detached from number : a length of a line, an area is a surface, a measure of an angle, are all numbers and we will see more involvement of number in abstract geometry.

The Ancients thought numbers are only integers and the integer parts of integers (*rationals*) , like

$$\text{Half } \left(\frac{1}{2}\right), \quad \text{third } \left(\frac{1}{3}\right), \quad \text{fifth } \left(\frac{1}{5}\right), \quad \text{two fifth} \left(\frac{2}{5}\right), \ldots .$$

Let AB be *any* unit segment

Fig. Any segment whatever its length is, can be seen as a unit.

We try to divide it in equal parts, thus we obtain the rational numbers

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \cdots, \frac{1}{n}$$

Easily, by the use of a compass, one can divide a unit into an even number of equal parts, but to get an odd number of equal parts we do the following:

For example if we try to divide the above unit segment in 3 equal parts, and obtain the two rational numbers

$$\frac{1}{3}, \quad \frac{2}{3}$$

we construct first any arbitrary length r three times until we get an arbitrary point $\boldsymbol{C}$, then we draw two parallels to $\boldsymbol{CB}$ as in the figure

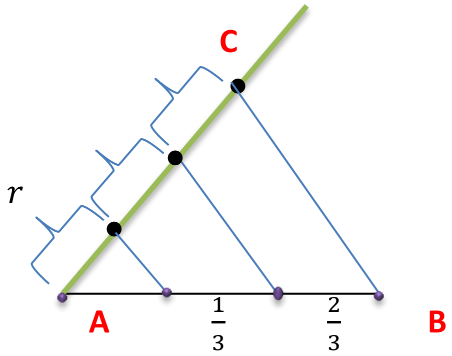

Fig. Construction of two rational numbers.

And we have divided the segment in three equal lengths.

Construction by revolution

As we have constructed some plane figures by appropriate tools, we can imagine another method to construct surfaces and volumes, it is a method that uses the rotation of geometric figures. The name *revolution* is taken from the Latin verb *revolvere* (to turn over) , so we give the following two definition:

Definition

$i)$ A *surface of revolution* is a surface created by the *rotation* of a curve C around an axis.

$ii)$ A *solid of revolution* is a volume formed by the rotating of a surface S around an axis.

These two operations may give many figures like:

$*)$ **Cylinder:**

If we mean the extern surface of a cylinder, then it can be constructed by the rotation of a line l around an axis Δ, and if we mean the solid ,i.e. the inside volume of the cylinder, i.e. the solid, then it can be obtained by the rotation of a rectangle D around an axis Δ.

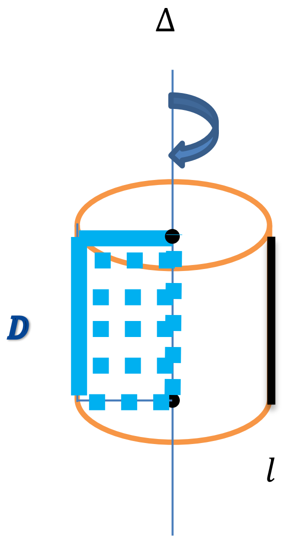

Fig. Construction of both surface and solid.

$*$) Sphere:

We can construct a sphere S by making a circle rotate around an axis Δ:

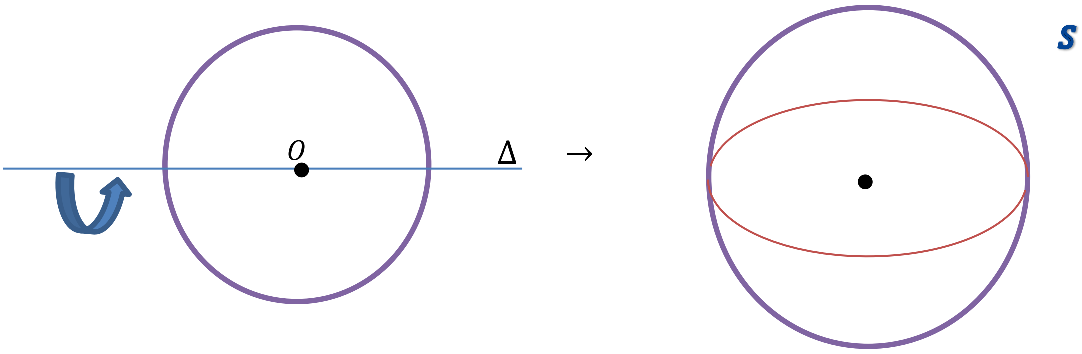

Fig. A circle turning around an axis passing by its center forms a sphere. Thus a sphere is different from *a ball*, a ball is constructed by turning a disk around the same axis.

$*$) Cone:

We get the surface of a cone by an oblique rotation of a line about an axis passing by a point p

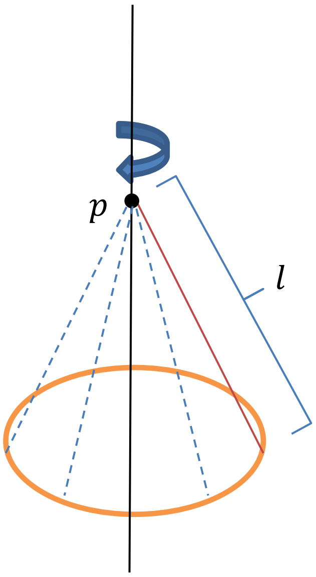

Fig. Extern surface of a cone.

The surface area of a cone and its base of the length l and radius r is

$$r\pi(l + r)$$

And the solid cone, by the rotation of a triangle around the vertical axis Δ:

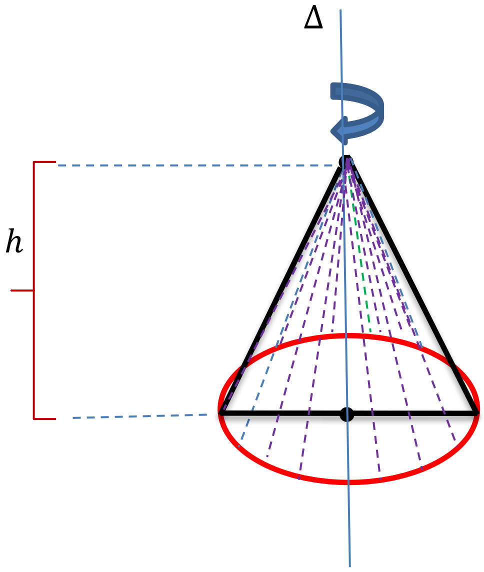

Fig. A solid cone.

The volume of a solid cone is:

$$\frac{1}{3}\pi r^2 h$$

where the factor πr^2 is the area of the base.

Pappus–Guldin theorems

In 1635 the Swiss mathematician Paul **Guldin**(1577-1643) stated and proved two theorems about surfaces and solids of revolution:

$a)$ The area of a surface generated by the rotating curve around an axis is the product of the curve length by the length of the path done by the curve geometric centroid $\boldsymbol{O}$

$$S = A_C \times l$$

b) The volume of revolution generated by the rotating surface is the product of the surface area value by the path length by its geometric center ***O***.

$$V = S_C \times l$$

But it is known now, that the Greek **Pappus** of Alexandria (29?-35? AD) also found such formulas before him, so the formulas are known today as ***Pappus*-*Guldin*** *theorems* .

Exercises

1) Draw a isosceles and equilateral triangles ,rhombus with a straightedge and a compass.
1) give all methods to section an angle in two equal angles(bisect an angle)
1) section an angle in 4 equal angles.
1) draw a line , and construct a parallel line to it with a straightedge , then with a compass
1) give all the methods to draw two perpendicular lines
1) give all the methods to draw the midpoint of a segment.
1) How can you draw two similar triangles?
1) Draw two parallel lines and a third intersect them in two points a and b. Indicate how many equal angles there are .
1) explain how can we construct the following solids by revolution:
i) oblique cylinder,
ii) half sphere
iii) two adjacent cones

1) Draw a circle with a cup, how can you find its center?

1) Is there a method to trisect a right angle by a compass? If yes, find it.

1) Construct with compass the first 3 rational numbers.

1) By the method of the book construct the first 10 rational numbers.

4

Geometry as a theory

Looking to the science of geometry through its history, there have been two totally different points of view that divided people in two groups: geometers and laymen. Otherwise said the points of view divided geometry in *Rational* Geometry and *Magic* or *Superstitious* Geometry. People in ancient civilizations were amazed and attracted by regular shapes, and thought about their possible magic meaning. They venerated the figures and related them to their beliefs by making them a part of local religions; some civilizations constructed altars and tombs in defined "sacred forms" , as did for example the Egyptians when they built the Big Pyramids to cover their kings' graves.

But geometers were more *rationalists* and had a different point of view. By curiosity, they felt the necessity of observing those geometric figures spending perhaps a lot of time, to discover their intrinsic nature. Making more deep observations this time, they passed to the stage of how to construct them and then how to calculate their areas and volumes. This activity led them to implement devices for practical purposes, and also to get useful mechanical constructions, as the defining of the center of gravity: an obscure link between geometry and the forces of earth.

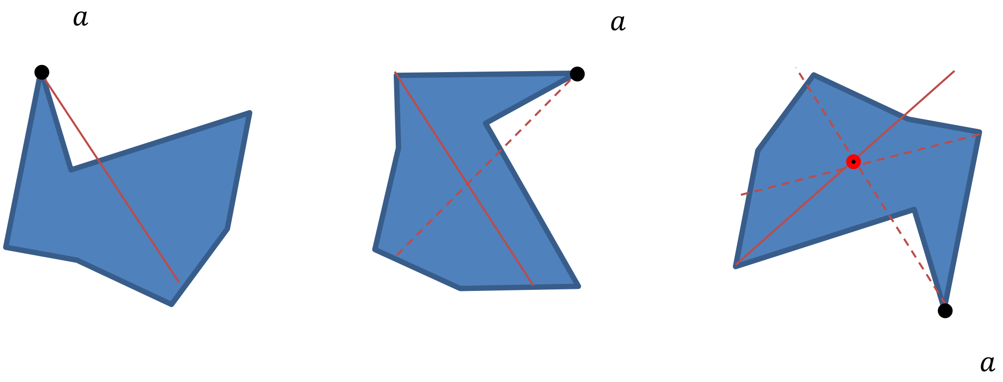

Fig. Center of gravity.

Then the next step has been the transition from *statements* and *observations* to the age of *proving proprieties* of geometric figures. Thus in this chapter, we will see the first rise of the *proof* concept. By remarking proprieties and relations inside and between geometric figures, and then proving them, we will make a system that gathers all specified *theorems* asserting those figures' proprieties; every system is called *a geometry theory.*

So we can put the preceding geometers' work in four stages:

$\ast$) Discover the figure and find its area or volume if it has.

$\ast$) Construct the figure only by a ruler and a compass if possible.

$\ast$) Observe more interesting properties of the figure than area and volume.

$\ast$) Then comes the stage of proving these proprieties and making a geometric theory.

Since the Greeks played an essential role in the creation of this old science of geometry, and to understand well geometry evolution, we will give the order in time of some of the most important Greek mathematicians (geometers) that worked through a long period of 1000 years, by this we now ready to make an idea about Greek geometry:

Thales

Pythagoras of Samos

Eudoxus of Cnidus

Euclid

Apollonius of Perga

Archimedes

Heron

Pappus of Alexandria

Proclus

To be more lucid and to have an idea of where we are situated in all this, we present the long file in the following chart:

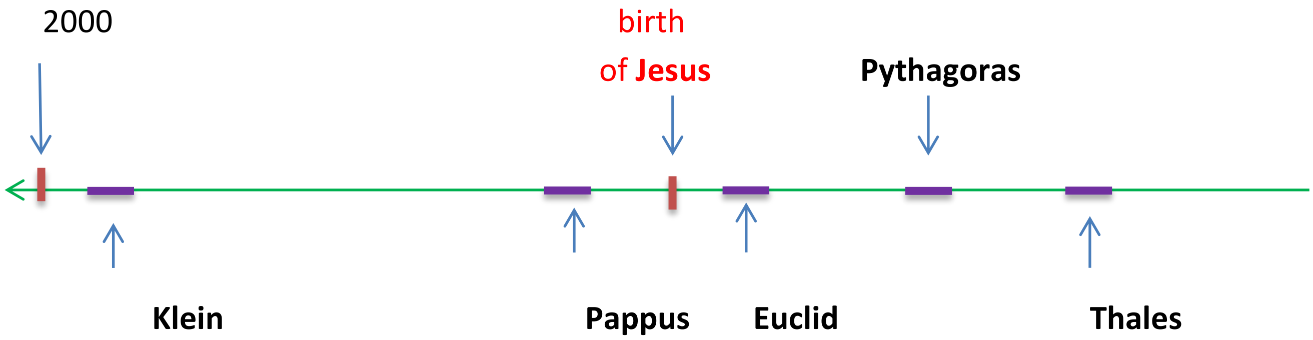

Fig. From **Thales** to **Klein**

Historians of mathematics consider **Archimedes** the greatest scientist of Antiquity. The activity place of these geometers was in the Mediterranean Sea. There was a great scientific activity in a town near the end of the Nile River named Alexandria (now in Egypt) ,which was the center of science in the world at the time.

The geometry of the Ancient Greeks is called now *coordinate-free* geometry[10], because it takes place in a plan or in three dimensions space without a link to a frame, that we call today *coordinates axes*. This has been the case with all their successors through ages, and also after the fall of Rome in 345 A.D., until the introduction of coordinates by **Descartes** and **Fermat** at the threshold of the Renaissance.

Structure of a geometric theory

Even before **Euclid**'s axiomatic method, we know that geometers formed a theory for geometric facts even if it was a little not systemized. After geometers define the objects of their study, they form thus a theory; it is the collection of *true facts* about geometric figures. These facts, proved by logic, are:

Lemmas, propositions, corollaries.

to all of which , we can simply give the name *theorems*.

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[10] _ Called also : synthetic geometry.

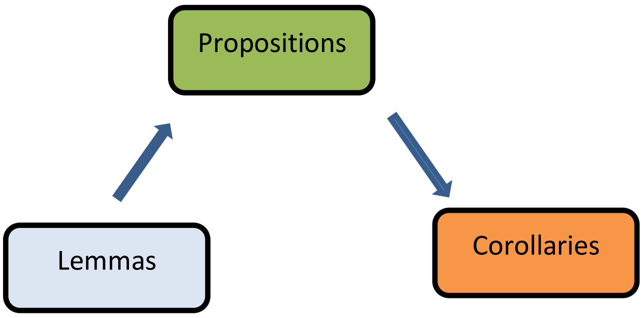


Fig. Conventionally, there are three categories of theorems, depending on their importance.

To give a classical view of what the components of a geometric theory are, we follow the French mathematicians **Legendre**(1752-1833) who wrote in his famous book on geometry (p.10):

Theorem: is a truth which becomes evident with a reasoning called *proof*.

Lemma: is a truth used alternatively to prove a theorem or to solve a problem.

Proposition: common name for theorems, problems, lemmas.

Corollary: is a consequence that results from one or many propositions.

Scholie[11]: is a remark on one or many propositions aiming at defining their relationship, their usefulness, restriction, or extension.

Hypothesis: a supposition done either in a statement of a proposition, or inside the proof.

---

[11] In his book on set theory, 20th **Bourbaki** group wrote:" *A commentary on a particularly important theorem appears occasionally under the name of "scholium""*. See also **Dieudonné** (p.26)

Today the **Legendre** terminology learned from preceded geometers is much followed. A proved fact, whatever its importance is, is called a theorem, and the collections of all theorems (lemmas, propositions, corollaries) form a theory.

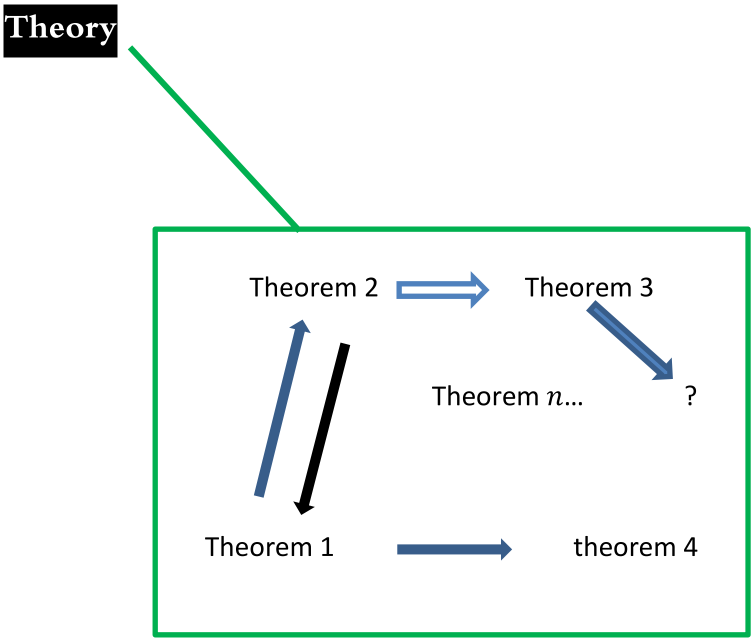


Fig. A sketch of a geometric theory.

Methods of proof

In geometry, one can be sure enough of a ***geometric truth***, and can convince himself of it, by constructing a drawing, but to ***prove*** a geometric statement or disprove it is another thing , and geometers have for ages used the same methods of proof:

a) Find a counter-example

To disprove the statement ” all perfect figures on the plane are convex ” a geometer can point out that the circle is a plane perfect figures but it is not convex.

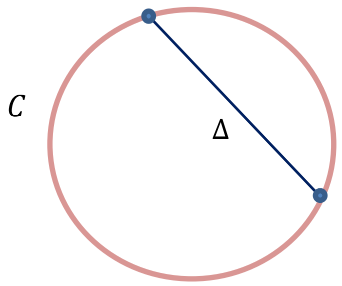


Fig. The circle C is not convex because not all the points of the segment Δ belong to C.

b) Prove directly

By the implication :

$$P \Longrightarrow Q$$

read “ if P then Q ”.

To prove the statement ” in any rectangle triangle : $\alpha + \beta = 90$ ” , a geometer reasons like this:

The sum of a triangle’s angles is 180^o, so we have

$$\alpha + \beta + 90 = 180^o$$

this gives

$$\Longrightarrow \alpha + \beta = 180^o - 90 = 90. \blacksquare$$

$c)$ Proof by contraposition

By the equivalence :

$$(P \Longrightarrow Q) \quad \Leftrightarrow \quad (\neg Q \Longrightarrow \neg P)$$

It is usually easier to give a proof of a statement by using contraposition then to prove it directly. To prove the statement:

"in an equilateral triangle all the angles are the same"

a geometer reasons like this:

Suppose that two angles are different and we have for the three sides $a = b = c$, then the sin law does not work and

$$sin(\alpha) \neq sin(\beta)$$

which means that $\neg Q$ is false, proving the statement. ■

$d)$ Proof by *reductio ad absurdum*

By the implication :

$$(\neg P \Longrightarrow \perp) \Longrightarrow P$$

which means : to prove P , we suppose that P is false and we are lead to a contradiction, which means that P cannot be false.

We give the following example:

Given the two lines Δ_1 and Δ_2 such that $\Delta_1 \parallel \Delta_2$ and $\Delta ⊾ \Delta_1$. Show that $\Delta ⊾ \Delta_2$.

To begin, we suppose that Δ is not perpendicular to Δ_2 . Since Δ makes an angle of 90^o with Δ_1 and the two Δ_1 , Δ_2 are parallel , than the complementary angles are equals to 90^o , but we have supposed that the angle α to Δ_2 is

$$\alpha \neq 90^o$$

a contradiction. ■

Thales

The first Greek geometer in our line was **Thales** (VI century BC) who was the very early recorded mathematician. He was born in Miletus (in nowadays Turkey), it is very known that he has much learned from Egyptians and Babylonians. It is believed that both the *proportions* and *Pythagoras* theorems were known before him.
Thales is known to us by two results in plane geometry:

First Thales' Theorem

Let C a circle (or a disc) and KL its diameter, M is any a point on the circle different from K and L.

Then KLM is a triangle right in M .

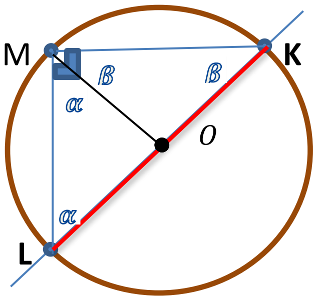


Fig. **Thales'** theorem

Proof:

First, we construct the segment MO, thus constructing two isosceles triangles which we know that each one has two equal angles α and β. But we know also previously that

$$2\alpha + 2\beta = 180$$

which proves that $\alpha + \beta = 90$ ■

The reciprocal is also true: in any right triangle KLM, with hypotenuse KL we can draw a circle C of diameter KL such that the point M is a point on the circle C.

We now move to the second theorem attributed to **Thales**:

Second Thales' theorem

Let ABC a triangle and two points , E : the point D on AB and E on AC. Then we have the equalities

$$\frac{AB}{AD}=\frac{AC}{AE}=\frac{BC}{DE}$$

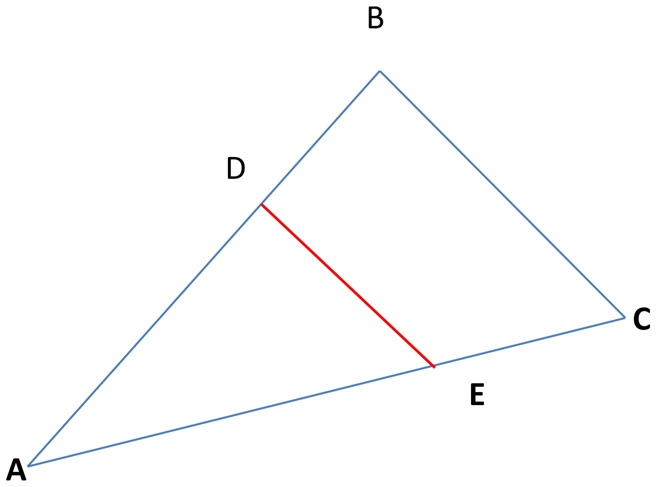


Fig. Second **Thales**' theorem, since DE is parallel to BC then $AB/AD = AC/AE$.

it is also called the *intercept theorem*. From this theorem we can construct figures that are called : *similar* figures.

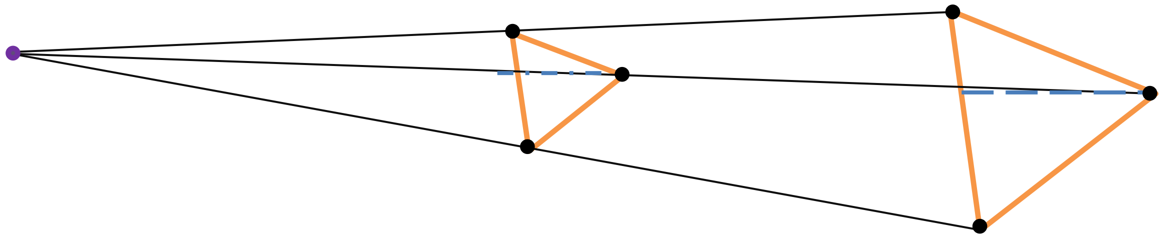


Fig. Two proportional, and thus similar, triangles.

Legend says that by implementing his result, **Thales** could find the height of one of Egypt's pyramids, his idea and method are well illustrated by the following drawing:

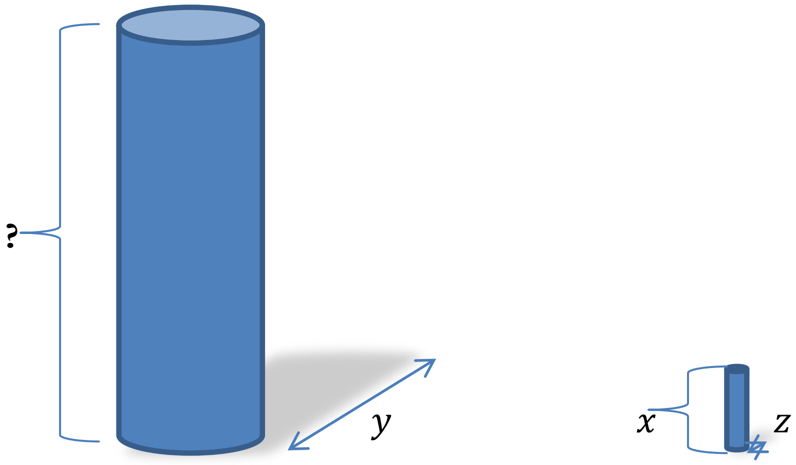


Fig. From **Thales** : $\frac{?}{y} = \frac{x}{z}$

Pythagoras' Theorem

Pythagoras was a very influential geometer and a lot of results in Greek mathematics before **Euclid**, are due to his school. Unquestionably, to the name **Pythagoras** is linked the very famous theorem relating the three sides of any right triangle.

Theorem:

Let ABC a right triangle, then we have

$$AB^2 + AC^2 = CB^2$$

As before, by $AB, \dots$ we mean the lengths : $\overline{AB}$,…

Proof:

The proof of this result for any square is quite easy. Here, we give a proof for general rectangles.

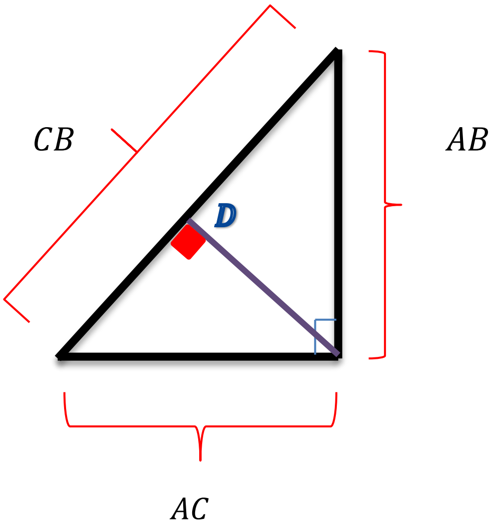


Fig. A right triangle with **Pythagoras** theorem.

Let ABC a right triangle in A, we draw the perpendicular AD that pass by BC at D, then the triangles ABC, ADC and ABD are similar, which via **Thales** theorem of proportions we get

$$\frac{AB}{BD} = \frac{BC}{AB} \qquad and \qquad \frac{AC}{DC} = \frac{BC}{AC}$$

or

$$AB^2 = BC \times BD \qquad and \qquad AC^2 = DC \times BC$$

and since $BD + BC = BC$ then :

$$AB^2 + AC^2 = BC^2 \qquad \blacksquare$$

And the inverse is also true, it is called:

Pythagoras inverse theorem

Let ABC a triangle such that

$$AB^2 + AC^2 = CB^2$$

then ABC is a right triangle in A.

Which gives an equivalence, or in modern notation an *if and only if* relation, written " iff " , otherwise :

*The **Pythagorean** propriety is a necessary and sufficient condition for a triangle to be right.*

Thus by modern notation:

Theorem

$$AB^2 + AC^2 = CB^2 \quad \Leftrightarrow \quad ABC \text{ is right triangle}$$

This theorem has many applications far from plane geometry, for example in the theory of numbers, we search for rational numbers (including integers) (a, b, c) that verify the identity

$$a^2 + b^2 = c^2$$

these are called **Pythagorean** triplets, we know that there are infinitely many of them, here are three examples :

$$(3,4,5), (6,8,10), (7,24,25)$$

The generalization of **Pythagoras** theorem for any exponent $n > 2$ is not true and has given the very known **Fermat** Last Theorem, that there are no integers a, b and c such that

$$a^n + b^n = c^n$$

when $n > 2$.

There exists also another equality, which in fact may be considered as a generalization of **Pythagoras** theorem to any triangle not necessary right. Exactly in any parallelogram $ABCD$, we can prove the *parallelogram equality*

$$2AB^2 + 2AC^2 = CB^2 + AD^2$$

We will see it when we use vectors.

Incommensurable numbers

Studying geometry, we have already met numbers like integers, and then we discovered as did the Greeks that with integers we can form new numbers: they are the rationals , and today we know other more theoretical than practical numbers like : the algebraic and transcendental numbers. As we have said the situation during the Greek geometry era was different. Before the time of **Euclid**, the **Pythagoreans**, or school of **Pythagoras**, stated that *all is number*, and they believed in two numbers only:

$i)$ The integers: very used in their common life, they are

$$1, 2, 3, 4, 5, 6, \dots$$

Constituting what we name today the positive natural numbers $\mathbb{N}^*$, of course there is no evidence[12] that the Greeks knew the (abstract) number 0.

$ii)$ The rationals: made by the combination of the integers :

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \dots$$

used also in their common life (half of a loaf, third of a loan, …) . This infinite set is the set of rational numbers $\mathbb{Q}$, so

---

[12] In fact the Greeks knew all the real numbers except the number 0.

$$\mathbb{N} \subset \mathbb{Q} \subset \mathbb{R}$$

and it seemed that the real line is composed only by rational numbers. It was through geometry of triangles, that the Greeks made a great discovery, namely that there exist other very strange quantities: *the incommensurable numbers*, or what we name today the *irrational* numbers, and from the point of view of integer numbers there is a dichotomy

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c = \mathbb{Q} \cup \mathbb{I}$$

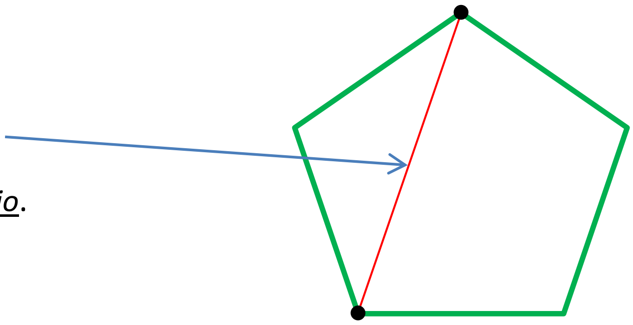
***Theorem*:**

Let ABC a right triangle, both the segments AB and BC are of length 1(unity), than AC is incommensurable with AB and BC.

Otherwise, we cannot measure B , BC and AC with the *common measure* , thus the word *incommensurable* took its name. This also means that the number $\sqrt{2}$ is not rational, we cannot find two integers a, b such that

$$\sqrt{2} = \frac{a}{b}$$

Another classical example of these incommensurable numbers is the *golden ratio*, the number measuring the diagonal of a pentagon:



The length of this segment
is an irrational number
it is the famous *golden ratio*.

Fig. A *pentagon* and its diagonal in red.

Exercises

1) prove the sin laws anew by another method.

1) show that in a parallelogram every two angles are equal

1) show that the two axis of a rhombus are perpendicular.

1) in an equilateral triangle ABC show that :

i) the bisector of an angle is perpendicular to the segment

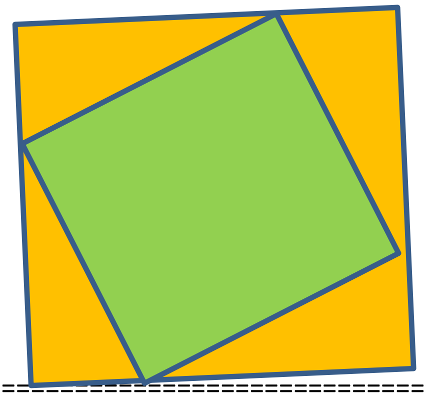
ii) the bisector bisect the segment.

1) infer from the intercept theorem seen above that :

$$\frac{AB}{BC}=\frac{AD}{DE}$$

1) show that he angle of a equilateral triangle are equal. What is its value?

1) draw the theorem of **Pythagoras** geometrically and get the proof :



1) prove **Pythagoras** theorem for a square of side a.

1) prove **Pythagoras** theorem for any rectangle by using cos and sin.

1) could you prove it geometrically by the same method for a rectangle of sides a and b ? where is the problem?(**Cavalieri**)

1) prove it by another method(see the previous chapters)

1) ABCD is a rectangle of a and b, add $\frac{a}{b}+\frac{b}{a}$, what do you remark?

1) Using **Pythagoras** theorem , show that $\cos^2(\alpha)+\sin^2(\alpha)=1$

1) prove that two lines perpendicular to the same line are parallel.

1) let C_1, C_2 be two circles with centers O_1, O_2 that intersect in a and b prove that segment $[ab]$ is perpendicular to the segment $[O_1O_2]$.

1) show that the three points form an equilateral triangle.

1) show how to find the height of a building from its shadow(second Thales' theorem)

1) find the formula for the length of a chord x

1) draw aright triangle with $(a,1,1)$ measure its hypotenuse, then show that the number a is not rational

1) can you an angle in 6 equal angles by a compass? Why?

1) show that a line passing of the middle of a,b is parallel to the third c.

1) show that the c is halve of the d (use sin law or proportions)

1) use **Thales** proportions theorem to prove the result.

1) according to you: what the essential propriety that makes **Thales** first theorem true?

1) take any unit you want , is the contour of a triangle always greater, lesser or equal to its area?

1) Show that the golden ration is irrational(use **Pythagoras**).

5

The Elements of Euclid

Until now, we have dived in classical geometry enough to understand what is going on, we have introduced proof methods , and proved some very good and easy! results in geometry by using logic. We now will discover a new strategy to treat this science, and which has been introduced by the Greek geometer **Euclid** in his monumental work : *The Elements*.

In fact, the *Elements* is not properly a book of geometry only, it contains the mathematics of those days: theory of numbers (preferably called *arithmetic*) and plane geometry. It is composed of 13 parts, each part is called a Book, and its contents can be presented as follows:

Books I, II, III, IV, VI XI, XII, XIII : geometry.

Books V, VII, VIII, IX, X : arithmetic.

The dates of **Euclid** are 330-275 BC, he was working in Alexandria. In his book, especially for geometry, **Euclid** devised a masterpiece, a new methodology to the treatment of the old science.

The *Elements of **Euclid*** has been considered through ages as a mathematical miracle, the *Holybook* of Geometry. It is still very amazing how a person could give such original work during very remote time, when humans were at their primitive scientific thinking, not having even a mathematical notation. More amazing is that his method remains even today after more than 2000 years a source of inspiration to scientists in general. About his work, we can read in many references many words of praise such as:

> ***Euclid**'s Elements is by far the famous mathematical work of classical antiquity, and also has the distinction of being the world's oldest continuously used mathematical textbook.*

also

> ***Euclid**'s Elements constitutes one of the great success stories of world literature. Scarcely any other book save the Bible has been more widely circulated. Over a thousand editions of it have appeared since the first printed version of 1482, and before its printing, manuscripts copies dominated much of the teaching of mathematics in Western Europe.*

The aim of the *Elements* regarding geometry consists in proving geometric proprieties known to the big Greek geometer and that by using some preparatory ground; by this in fact, **Euclid** has given the first *foundational* work in mathematics. He treated the geometry of his days in a new systematic way by following steps:

Definitions.

Postulates.

Axioms (or : Common notions).

Methods of proof.

The proof of proprieties of figures by logic.

These steps on which the geometer will build on to prove geometric properties, has remained a paradigm for mathematical systems until this time, and to understand **Euclid** work we shall now take a close look at some of its aspects.

The Definitions

They are 23 explanation of the geometry objects …studied in the *Elements*, presented to the reader by plain natural language, among them:

$i)$ Point.

$ii)$ Straight line.

$iii)$ Curved line.

$iv)$ Parallel lines.

$v)$ Surface.

$vi)$ Angle, obtuse angle, acute angle.

$vii)$ Circle.

The postulates

These are self-evident truths, which are considered as *non-provable* and necessary theorems that **Euclid** *demanded* from the reader to take for granted, possibly because he himself could not prove them and at the same time felt that no one can prove them either. We see that the first three ones are geometric *constructions*, the fourth is an *observation* and the last one is of a different nature:

$i)$ We can draw(construct) a line(a segment) between two points.

$ii)$ We can draw an infinite line.

$iii)$ We can draw a circle of any size.

$iv)$ all right angles are equals

$v)$ the famous 5th postulate that we shall see later separately.

Axioms

The following are proprieties that **Euclid** called *Axioms*[13], which are simple facts known to the modern reader and we have seen and used some of them already. They are also called *Common Notions* , as the reader will see they are almost about relation between *quantities*:

$i)$

$$(A = B \quad and \quad B = C) \Longrightarrow C = A$$

$ii)$

---

[13] The oldest copy of *The Elements* was due to geometer **Theon** of Alexandria(330-400). It is important to note that since his time, and in the different copies of *the Elements* that circulated in the Middle Ages, some postulates were considered as axioms, and vice versa, other axioms were considered as postulates.

$$A = B \quad then \quad A + C = B + C$$

$iii)$

$$A - C = B - C \;\; then \quad A = B$$

$iv)$

$$A \neq B \quad then \quad A + C \neq B + C$$

$v)$

$$(A = 2C \quad and \quad B = 2C) \Longrightarrow A = B$$

$vi)$

$$(A = \frac{1}{2}C \quad and \quad B = \frac{1}{2}C) \Longrightarrow A = B$$

$vii)$

if $F \subset E$, then : measure$(E) >$ measure (F).

Method of proof:

We will not see new methods of proof, we have already sited them in the previous chapter, but we will use a new methodology to implement these methods. The new thing is to start the proof by considering definitions and axioms, then by building on them and using classical common logic we establish the geometric truth.

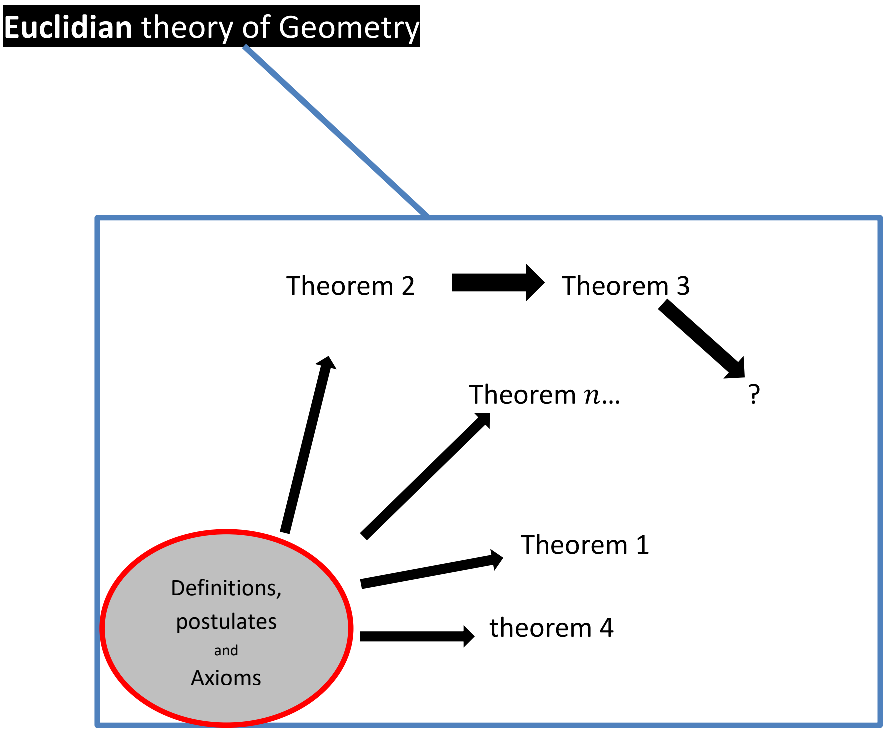


Fig. Axioms are a basis of any **Euclidian** theory.

So in this new edifice settled by **Euclid**, we prove a geometric statement rigorously in a new elegant way very appreciated and exclusively used by modern mathematics; these proofs are called *synthetic proofs*. Later in analytic geometry we will prove statements by the use of coordinates; in this case the proof is called *analytic*.

The theorems of the Elements

They are the results proved by **Euclid** in his book *The Elements*. There are a lot of them, so we expose some of the known ones to the reader since pre-university school, and we have seen already some of them in the previous chapters:

Proposition 6:

If a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another.

This triangle is an isosceles triangle, it is a triangle with two equal angles and sides. And a proposition about measurements:

Proposition 20:

In a triangle, the length of any two segments is always greater than of the third.

This is now known as the *triangular inequality* and it is very used between positive numbers in analysis. And now another result (who came later):

Proposition 47:

In right angled triangles, the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.

Of course this is the celebrated **Pythagorean** theorem, a very important result in geometry and analysis of which we have already seen the proof.

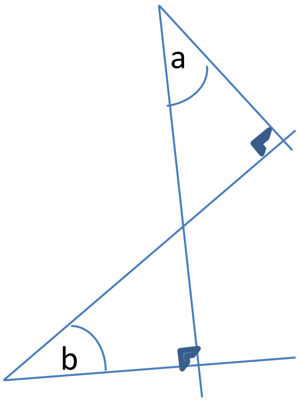
Exercises

1) Give the definition of your own

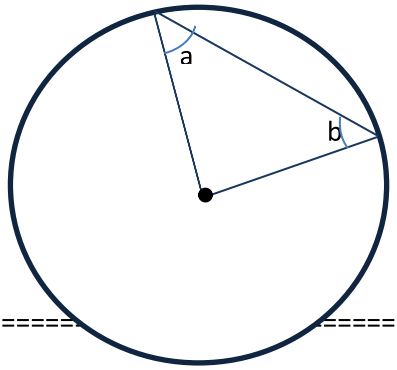
-point
-line
-plan

1) what do you think of this definition: ” two lines are parallel of they do not intersect in a unique point ”?

1) prove that $a = b$



1) In the bellow drawing, prove that $a = b$



1) By the use of the proprieties of the interior angles, show that the **Euclid**'s fifth axiom is equivalent to the fact that the sum of the angles of a triangle is 180^o.

6

Euclidian Flaws

We do not know if **Euclid** after he had constructed his *Elements* intended, as any author after all, to fix things in his reasoning and to put his geometric work in a more coherent one. Anyway, Greek geometers and their followers have been amazed by **Euclid**'s masterpiece and through studying it, they tried to bring remedy to its flaws.

About the contents of the *Elements* one can make some observations:

—It is not evident, what the difference between an *axiom* and a *postulate* is. So what is the difference between the *postulate*

All right-angles are equal to one another

And the *axiom*

things equal to the same thing are also equal to one another?

Whatever the reason for which **Euclid** made a difference between *axioms* and *postulates*, nowadays mathematicians make no differences between both the terms and the concepts, so :

In what follows we will not use the word "postulate" any more, we use only the word "axiom".

– The postulates given by **Euclid** are not of the same nature and complexity.

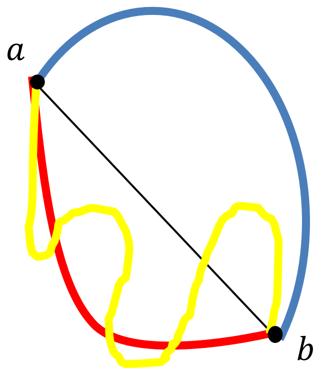
–Some axioms and postulates can be proved, and cannot be considered as non-provable statements that we demand the reader to take for granted. The very famous that we will see later : the fifth axiom.

–**Euclid** did not give any definition to many notions he used : surface, volume, movement,…

Archimedes

Archimedes tried to renew and promote **Euclid** system of geometry, so he could add some "axioms":

$i)$ A segment is the shortest lines between two points a and b.



$ii)$ **Archimedes** axiom: if $x > y$ then there is a number n such that

$$\alpha x < y$$

At the time this was seen by length of segments

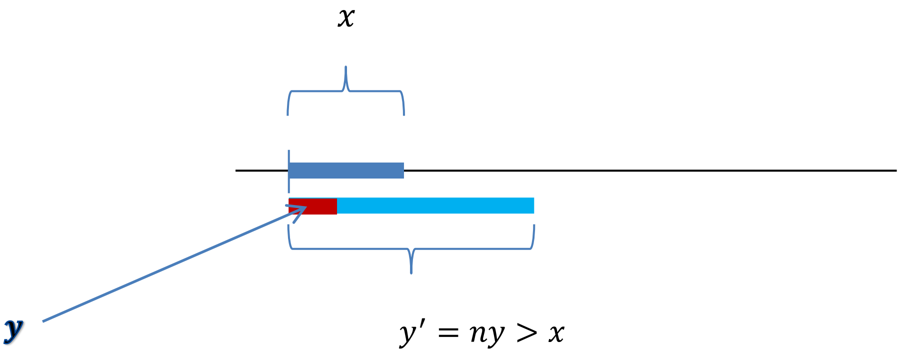


Fig. The **Archimedes**' axiom.

$iii)$ The plane is the smallest surface having the same plane boundary.

==

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University of Constantine

However, all of **Archimedes** axioms[14] except one are superfluous and can be proved, and he like **Euclid** also did not give definitions for notions he used as : length, volume and surface.

Axiomatic method:

Euclid method to treat logically the science of geometry, have been very influential through ages, and I want here to insist on the fact it has become a way of thinking not only in mathematics , but in other disciplines other than geometry. This is the case at such an extent that German mathematician **Hilbert**(1862-1943) in his 24 problems addressed in 1900 to mathematical community , proposed the problem 12 as follows :

Can we find an axiomatization to physics[15]*?*

In one of the physics books written in the XX century, we can read the following:

> *An axiomatic development of mechanic, as for any science, should contain the following basic ingredients:*
>
> 1. *Undefined terms or concepts. This is clearly necessary since ultimately any definition must be based on something which remains undefined.*
>
> 2. *Unproved assertions. These are fundamental statements, usually in mathematical form, which it is hoped will lead to valid descriptions of phenomena under study. In general these statements, called axioms or postulates, are based on experimental observations or abstracted from them. In such case they are often called laws.*
>
> 3. *Defined terms or concepts. These definitions are given by using the undefined terms and concepts.*

---

[14] **Pappus** too, has added other useless axioms to the **Euclid** system of geometry.

[15] The reader interested in this subject can return to **Corry**'s article : *David **Hilbert** and the Axiomatization of Physics* (1894-1905).

4. Proved assertions. These are often called theorems and are proved from the definitions and axioms.

An example of the" axiomatic way of thinking" is provided by Euclidian geometry in which point and line are undefined concepts.

What is an axiom?

Axiomatic thinking in the spirit of **Euclidian** geometry has its beginning in the XIX century, through many other branches of mathematics. It has been advocated that to build theories in modern mathematics, axioms are a necessity. But to be more precise, the word "axiom" in modern terminology has two meanings:

$i)$ A truth that is quite evident and we cannot prove in the same manner as the fifth axiom of **Euclid**. As **Legendre** says in his famous *Element of geometry* (1898) *(p.5)*

An axiom is a proposition which is evident by itself.

As for **Euclid** who could not make geometry without his 5th axiom, it seems that we cannot make good mathematics without axioms, one of the very famous and essential axioms in the whole of mathematics is the *axiom of choice*, which no mathematician since 1905 could neither prove nor abandon.

$ii)$ A propriety that the law of composition " $\circ$ " in an algebraic structure $(E,\circ)$ must verify, like:

Axioms of a group.

Axioms of a ring.

Axioms of a field.

Axioms of algebraic geometry…

As an example of the use of the second meaning in modern books, we take the example the additive group of real numbers$(\mathbb{R},+)$, with the law of composition " $+$ " it has the following axioms:

$i)$ $x+y$ is in $\mathbb{R}$ for all x and y. ($+$ is internal).

$ii)$ $x+y=y+x$ (axiom of commutativity).

$iii)$ $e+x=x$ $(e=0)$ (e is the neutral element).

$iv)$ $x+(y+z)=(x+y)+z$ (axiom of associativity).

So $(\mathbb{R},+)$ is a commutative group, and to prove for example that

$$0+x=x$$

we use the axioms of this group

$*$ $0+x=x+0$ (Axiom of commutativity).

$*$ $x+0=x$ (Neutral element axiom).

$*$ $0+x=x$, by using the first and the second together and the result follows.

The champion and the promoter of this methodology, sometimes very excessively, has been the French **Bourbaki** group, influenced much by the German school of abstract algebra and **Hilbert** ideas.

Exercises

1) Prove that in an isosceles triangle, two sides are equal.

1) Prove **Euclid** interior angle theorem (use **Thales** theorem).

1) Can we consider logically, an axiom as a theorem?

1) to you, when does a mathematical propriety become an axiom?

1) So could the 5th axiom be a proved theorem some day?

7

The Fifth Axiom

The most famous axioms of **Euclid**, without doubt was the fifth[16] which says:

Statement:

Let Δ be a line that intersects two other lines Δ_1 and Δ_2,then if the sum of the interior angle is less the two right angles(i.e. 180^0) then the two lines Δ_1 and Δ_2 must intersect at some a point .

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[16] The interested reader will get good account on **Euclid**'s Elements and the history of the attempts to prove his 5th postulate in "*Higher Geometry*" (Mir-1980) , a book by Soviet geometer Nicholai **Efimov**(1910-1982).

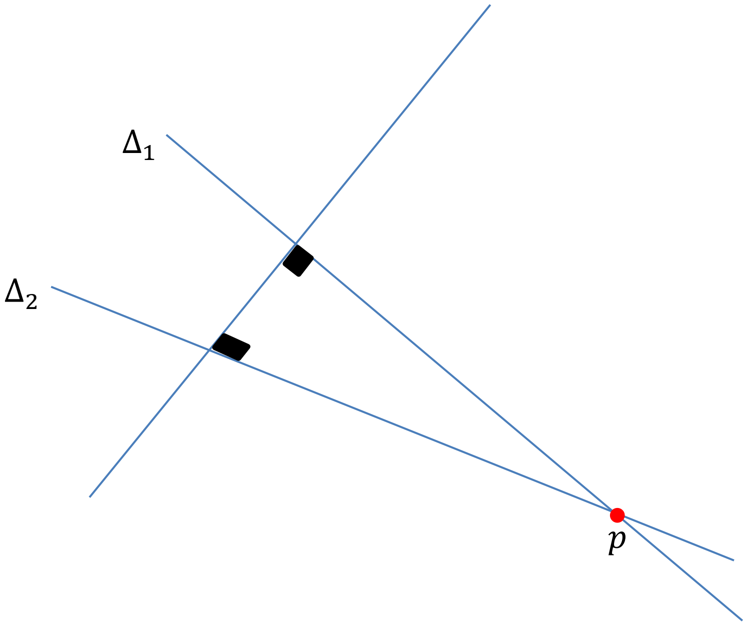

Fig. The 5th axiom .

Or from ancient Greek (Tr. **Heiberg-Fitzpatrick**):

> *If a straight line falling across two straight lines makes internal angles on the same side whose sum is less than two right angles then the two straight lines being produced to infinity meet on that side.*

Since the appearance of the *Elements* until the XIX century, the fifth axiom of **Euclid** has been the most controversial and vulnerable assertion in geometry.

What is wrong about the 5th axiom?

Question

Why did geometers thought it could be proved?

Evidently because the 5 postulate is more *complex* in nature and just not a postulate, it does not look like the other 3 postulates:

1 we can draw a straight line from any point to any point.

2 we can produce a finite straight line continuously in a straight line.

3 we can draw a circle with any center and radius.

which are constructive in nature, whereas the 5th needs a proof like for example the propriety:" *the sum of the angles in any triangle is two right angles*". It has also nothing to resemble the previous already seen *common notions* and the forth postulate:

4 all right angles are equal to one another.

1 things equal to the same thing are also equal to one another.

2 if equal things are added to equal things then the whole are equal.

3 if equal things are subtracted from equal things then the remainders are equal.

4 if equal things conceding with one another are equal to one another.

5 the whole is greater than the part.

In 1895, English geometer John **Playfair**(1748-1819) reformulated **Euclid** fifth axiom elegantly and more simply. In his book on geometry[17] , we read :

Playfair axiom:

Let Δ_1 be a line. From a point p, we can draw only one line Δ_2 parallel to Δ_1.

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[17] At p. 32, in the 1804 edition.

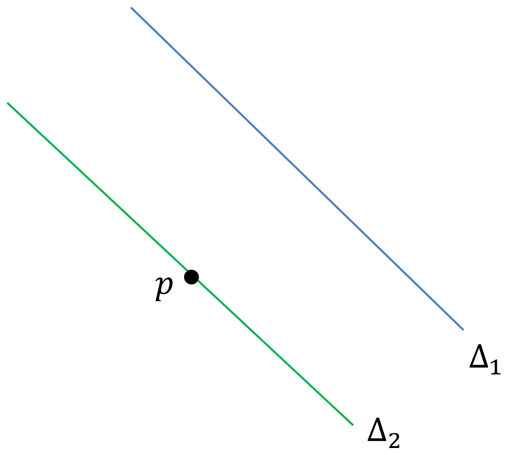

Fig. the 5th postulate reformulated by **Playfair**.

In order to feel in a sense how the fifth axiom is difficult to prove, the reader must know that geometers struggled with it for centuries, in any attempt to prove it without success. Here is an example of an attempt usually used: the known method *reducto ad absurdum*:

Suppose we have two lines Δ_1 *and* Δ_2 *passing through a point* p*, and they are both parallel to* Δ *, then the angles are different …*

but this result itself is a consequence of the axiom of the parallels, and this turns us to the same problem.

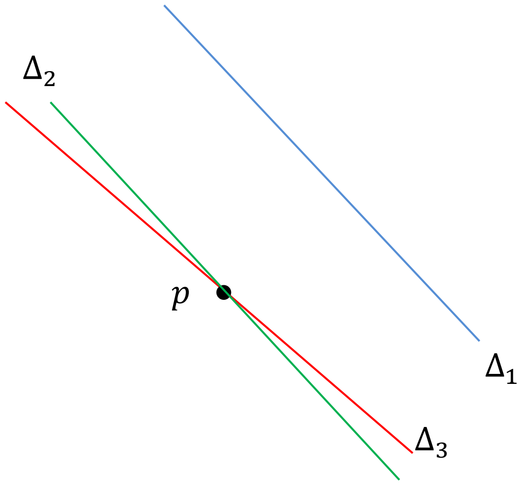

Fig. Negation of the 5th postulate.

Equivalence:

If two lines pass by a point p and are parallel then they superpose.

Geometers after **Euclid** attempted to prove the 5th axiom, and for any difficult problem in mathematics in all times, there was time that some Greek mathematicians believed they have proved it like **Proclus** (410-485 AD), one of the commentators of **Euclid**. During Islamic Middle Ages, and through reductions many Muslim geometers also tried to prove the axiom as did **Abhari**(?-1266).

At the time of the Renaissance there were also many attempts to prove it along with other Greek problems like the squaring the circle, and this continued until the introduction of non-**Euclidian** geometries as we will see later. So, we can here cite some mathematicians who tried to prove the old axiom:

−Italian Giovanni **Saccheri**(1667-1733) , his book published in 1733 .

−Swiss Johann **Lambert**(1728-1777) in his book(1766) " *Theorie Der pararallen* ", written in German.

−French **Legendre**(1752-1833) , he published many false *"proofs*" .

−Danish astronomer H.C. **Schumacher**(1780-1850) who sent a false proof of the axiom to **Gauss**.

And it turned out, as we will see, that the problematic was not to prove the fifth axiom but to find other axioms giving new geometries, and the fifth axiom is still an axiom for the **Euclidian** geometry.

Hilbert axioms for Euclidian geometry

After the appearance of non-**Euclidian** geometry, mathematicians of the XIX century felt the obligation to put geometry, including **Euclidian** geometry, on a solid foundation. And this foundation is based the following points:

i) Every axiom may not be deduced from another axiom, so the axioms should be *independent*, which means we cannot prove any of them from the other axioms.

ii) That means that the task is to find the minimum number of axioms that give all **Euclidian** geometry.

iii) We must get rid of intuition and experience, and this is an essential feature of modern mathematics, we will see and discuss this point later on.

Many sets of axioms were proposed, the very known[18] are those of **Hilbert** proposed in his book *Foundation of Geometry* (1899) , **Hilbert** has given a collection of axioms that presents a logical development of **Euclidian** geometry from the basics to the theorems.

Hilbert axioms are in fact 5 groups of axioms:

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[18] Other systems of axioms to found geometry other than **Hilbert's** exist, like and.

−Axioms of connection.

−Axioms of order, first studied by M. **Pasch**(1843-1930) in his book *Lectures on New Geometry*(1882).

−**Euclid**'s 5th axiom.

−Axioms of congruence.

−**Archimedes** axiom.

One of the much known axioms in **Hilbert**'s system is **Pasch**'s axiom.

Pasch's axiom (1882)

If a line does not pass by any of the points A, B, C in a triangle ABC , then if it intersects a segment of ABC, then it must also interests one of the others.

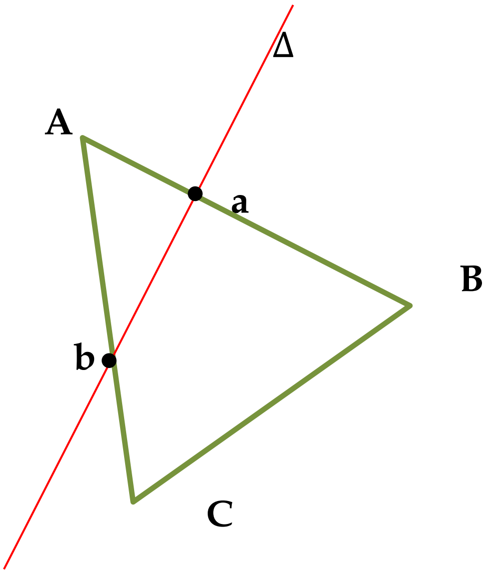

Fig. **Pasch's** axiom.

Exercises

1) Give the definition of your own of :

-surface

-area.

-volume

1) what do you think of these definitions:

i) two lines are parallel if they do not intersect in a unique point.

ii) a point in a plane is the intersection of all line passing by it.

8

Euclid's Legacy

The activity of geometers to give new theorems about geometric figures has not stopped after **Euclid**, so far, we have been working in the realm of ***Euclidian*** *plane and solid geometry,* and we will see in the subsequent chapters many transformations after **Euclid** breakthrough. But now we are still following the Greeks, so we return to remaining geometers in the order of time:

Apollonius of Perga.

Archimedes.

Heron.

Pappus of Alexandria.

Proclus.

Apollonius and the conics

We have already met the names of **Archimedes** and **Heron. Apollonius** (25?-19? BC) began his career studying at the **Euclidian** school. He was interested in a particular solid figure: the *cone*.

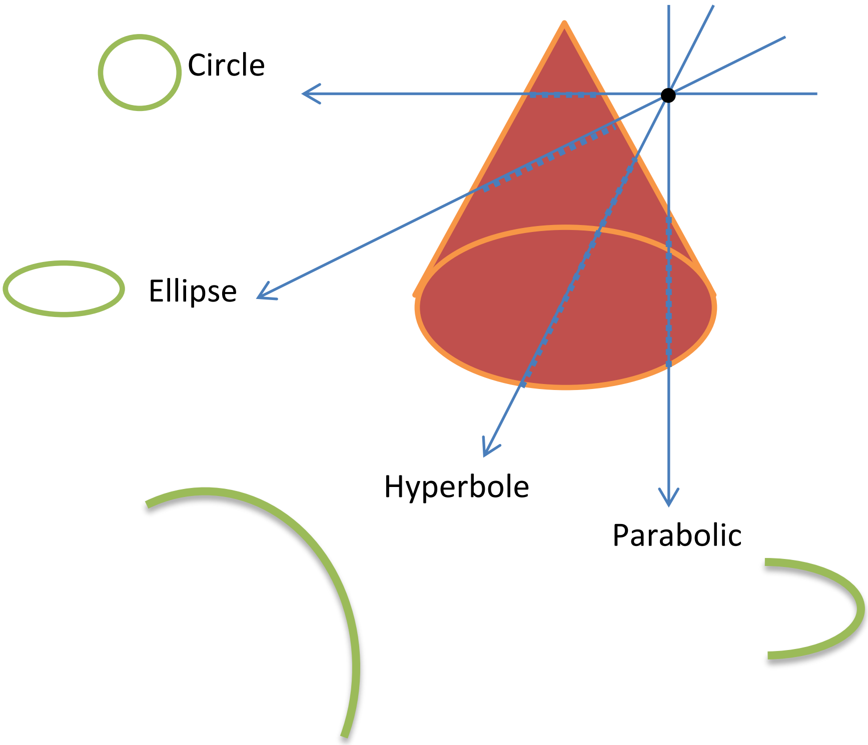

Fig. the 4 conic sections.

his aim, as any geometer, was to find new shapes, he discovered that when he makes the section (cutting) of a cone in four different angles he is able to obtain four different plane curves, they are the four *conics*:

The Circle: if the section is *parallel* to the base of the conic.

The Hyperbola : when the section is *parallel* to the one of the sides of the cone.

The Parabola: if the section is *perpendicular* to the base of the conic.

The Ellipse : if the section intersects the two sides of the cone but not parallel to the base.

So:

Definition:

A *conic* is a plane curve resulting from the cutting of a *cone* under a specified angle.

These cuttings are in $\mathbb{R}^3$ and the resulting curves can be seen either in $\mathbb{R}^3$ or in the plane $\mathbb{R}^2$. The conics are still ubiquitous today, but it seems that they were also known before **Apollonius** at the time of **Euclid**, so **Dieudonné**[19] thinks differently by saying

> *The invention of conics contrary to popular belief, which think they were invented by cutting a cone by a plane. This is completely wrong. The invention came though the solution of an algebraic problem. They wanted to solve what they called the double proportion:* $a/x = x/y = y/b$*. If you do this via algebra, you find at once it means finding the cube root of* a^2b*. And* ***Menechmus*** *student of* ***Eudoxus*** *around middle of the* 4th *century BC said "this is a curve" and invented the parabola, "this is another curve" and invented the hyperbola. And later it's only later that they discovered that this has to do something with sections of the cone.*

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[19] From his audio lecture held at the University of Wisconsin, on March 3, 1972.

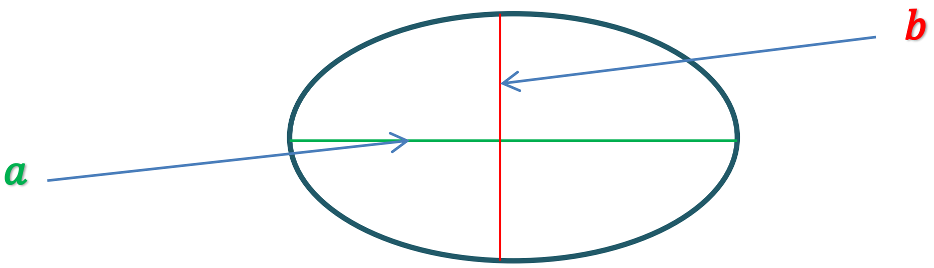

Fig. An ellipse.

We can calculate areas of two conics because they are closed, the circle and the ellipse, so the area of an ellipse is

$$\pi ab$$

We can also construct a *solid conic* by revolution around an axis and the resulting surfaces are:

i) by revolution the circle gives a *sphere* , which we have seen before.

ii) the hyperbole: gives a surface named the *hyperboloid*.

iii) the parabola gives a surface named the *paraboloid*.

iv) the ellipse gives a solid figure whose surface is named the *ellipsoid*.

These and others that we will see, are called *quadric* surfaces, and as solids they are not polyhedra because their sides are not plane. Particularly the ellipsoid was studied a lot from the XVIII century and through XIX because of its relation to the problem of gravity attraction (as Earth is an ellipsoid) and the **Newton** law of attraction

$$\vec{F} = G\frac{m_1 \times m_2}{r^2}\vec{\boldsymbol{e}}$$

Proclus(41?-485 AD) was implied in constructing geometric figures , including the construction of the four conics by other methods than sectioning the cone.

Fermat and the others

After all this, we can say that the Greeks are amazing after all, as they much discovered with very precarious means and methods. In the beginning of the Renaissance geometers got to know the geometry of the Ancients and *the Elements* from the Orient, we read in **Struik**

> *Mediaeval geometry was mostly taken from **Euclid**, whose work was partially known through the widely circulating works of **Boethius** (sixth century) and which became available through translations by Gerhard **Decremona**, Johannes **Campanus**, and others in the twelfth an thirteenth centuries. The first printed **Euclid**, in the Latin version of **Campanus** , was that of Erhart **Ratdolt** in Venice(1482) , a beautiful piece of work with many figures, and from that time on full or partial editions, in the original Greek and in translations, appeared in several countries.*

After the period of Middle Ages men of the Renaissance looked to the Ancients and their science with great esteem, translating and printing their books, charmed by their results. German **Clavius**(1538-1612) edited **Euclid** work and with lots of commentaries and he even added some axioms. Among other Europeans we can mention French mathematicians like : **Roberval**, **Fermat**, **Descartes**, **Pascal**, **Viète** ... as an example of their activity a little result due to **Pascal** about parallelism:

Pascal's parallelism theorem:

Let the six points in the following figure. If CB' is parallel to BC' and CA' is also parallel to AC', then BA' is parallel to AB'.

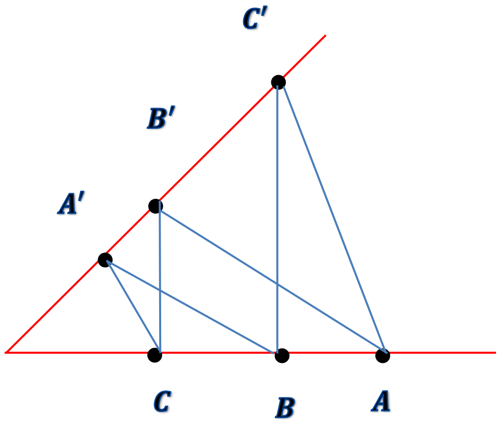

Fig. **Pascal**'s parallelism theorem

This theorem is among the theorems proved in **Hilbert**'s book *Geometry*.

Fermat has investigated a lot in classical mathematics, much inspired by the geometry of **Euclid**, he used **Thales** theorem to find similar figures. He was very active in finding *tangents* to many known curves, and this has helped much in inventing the calculus later.

One of the result due to him is his ***Fermat** point* in a triangle:

Definition

Let ABC a triangle and p is a point inside the triangle, the p is a **Fermat** point if the sum

$$Ap + Bp + Cp$$

is minimal.

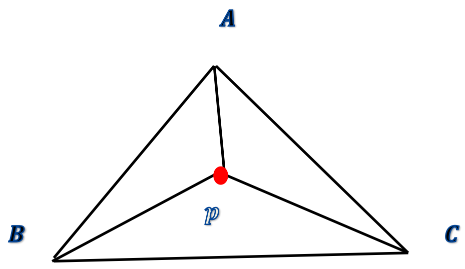

Fig. **Fermat** point.

Captivated by the theory of numbers, and while he was studying the work of the Greek **Diophantus** on number theory, he stated his ***Fermat** Last Theorem*, a problem that has given us much of abstract algebra and algebraic number theory. Once he said

> *"Perhaps posterity will be grateful to me for having shown that the Ancients did not know everything"*

In a reply to **Fermat** we cite here the words of American Harold **Edwards**, who wrote in 1975:

> *Nothing could show more dramatically the difference in attitude between **Fermat**'s time and our own. It is inconceivable that a twentieth century mathematician could think that the Ancients knew everything. On the contrary, in our times the general attitude is that, at least where mathematics is concerned, the Ancients knew nothing at all.*

This is really very true, because the Greeks had not even a mathematical notation to control more their science, and their geometry was mostly very practical much less than theoretical. As we will see the subsequent chapters, that the geometry of the Ancients will see many unimaginable transformations, we will see that **Fermat** himself will do many things in this area. He gave first steps into the road to modern geometry by pioneering analytic geometry; also he touched the two pillars of the calculus, first by the theory of tangents, considered as the geometric theory of differential calculus, and second by exhausting surfaces by rectangles, by which he prepared the way to integral calculus.

Exercises

1) Prove that any two segments of a triangle is greater of the third.
1) is it true that in a triangle ABC we have : $sin(\alpha) + sin(\beta) > sin(\gamma)$
1) what about cos?
1) $a + b > c$ then $ab = 0$
1) $a + b > c + d$ then $ab = 0$ and $0 \geq 0$
1) prove that the two angles in a isosceles triangle, are equal.
1) prove that two opposite angles are equal
1) prove that alternate angles are equal
1) prove that the sum angles of a triangle is 180^0 (straight angle)